Latest Advances in Spectral Linear Algebra in Network Science

Leo Torres and Tina Eliassi-Rad
Northeastern University

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The scientific, computational, and mathematical study of real-life systems that can be represented as graphs.

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offline social networks

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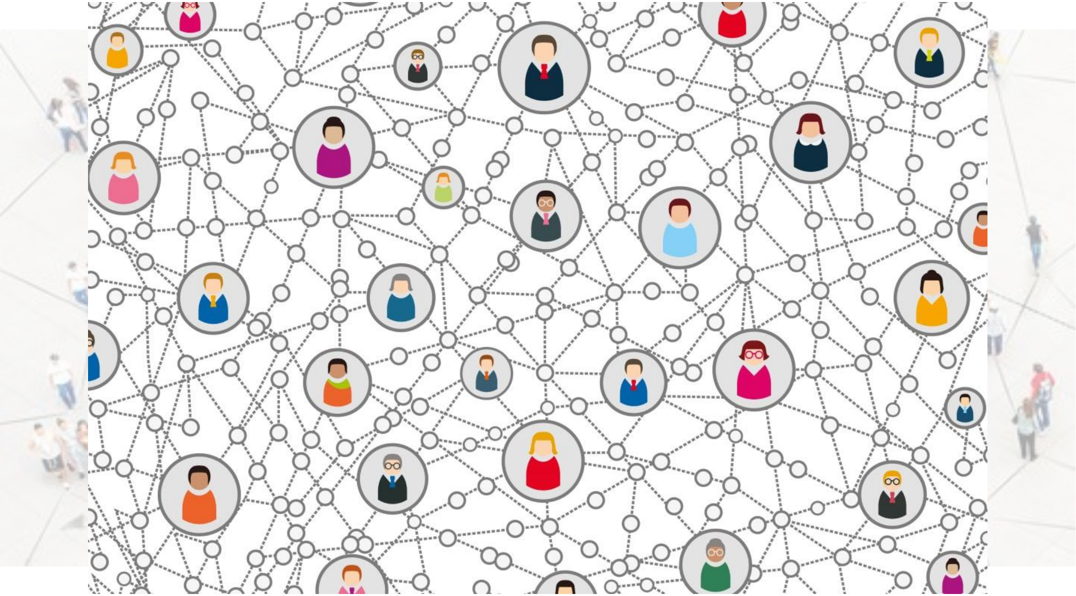
The scientific, computational, and mathematical study of real-life systems that can be represented as graphs.

offline social networks
online social networks



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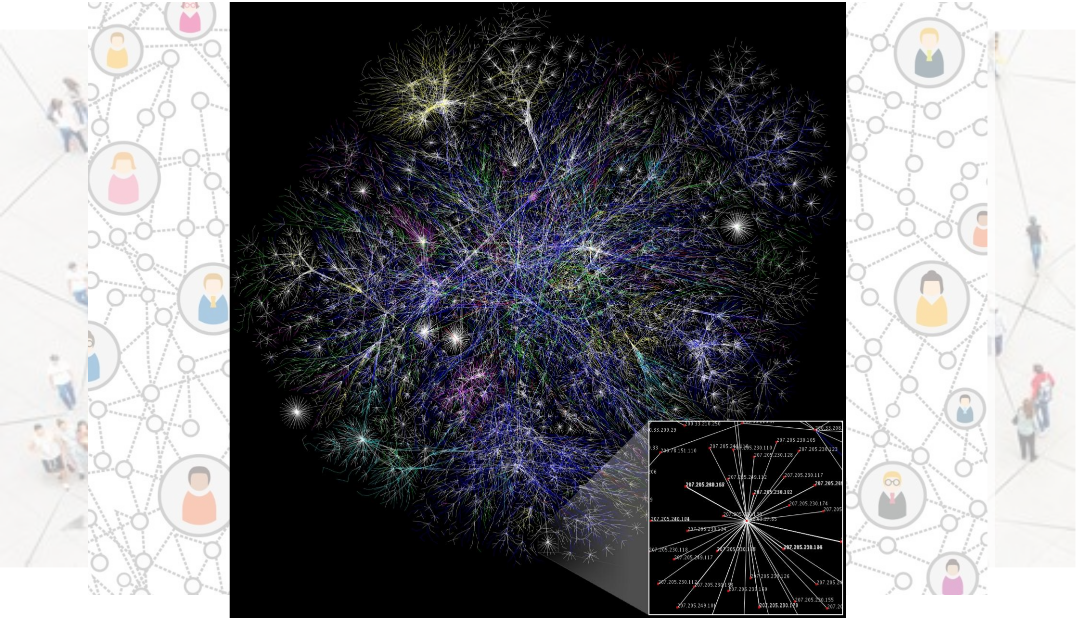
The scientific, computational, and mathematical study of real-life systems that can be represented as graphs.

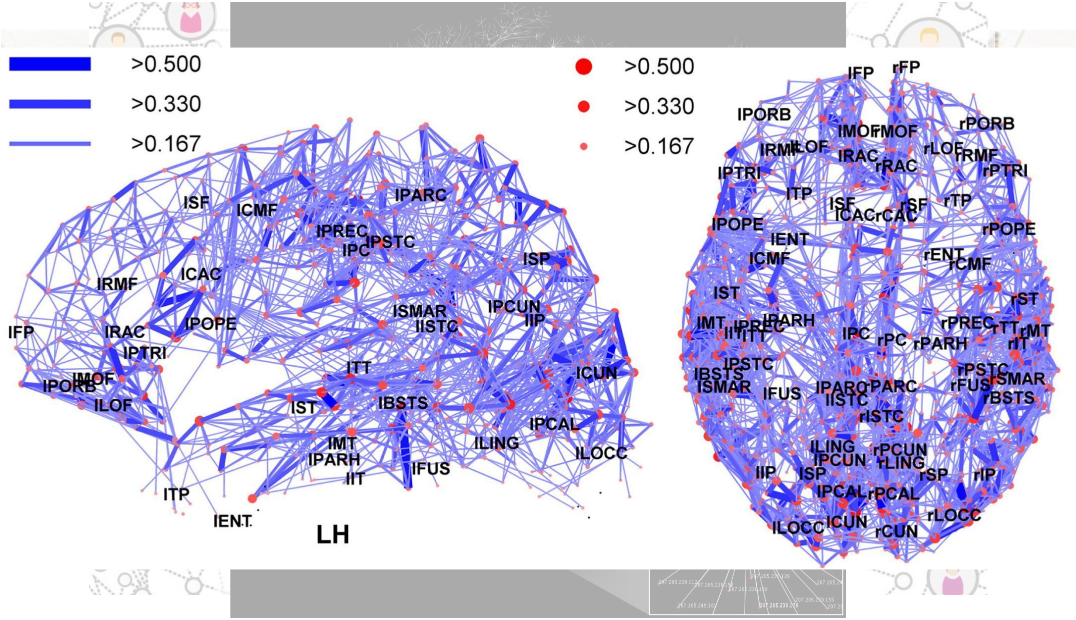
offline social networks
online social networks
internet devices



Latest Advances in Spectral Linear Algebra in **Network Science**

The scientific, computational, and mathematical study of real-life systems that can be represented as graphs.

offline social networks
online social networks
internet devices
brain networks



Latest Advances in Spectral Linear Algebra in **Network Science**

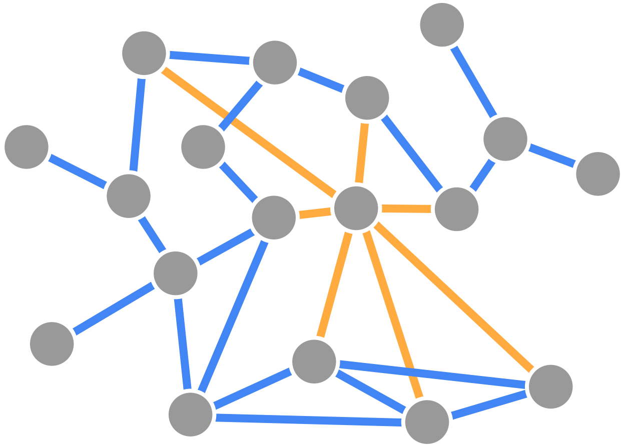
The scientific, computational, and mathematical study of real-life systems that can be represented as graphs.

offline social networks
online social networks
internet devices
brain networks
...

...and more!

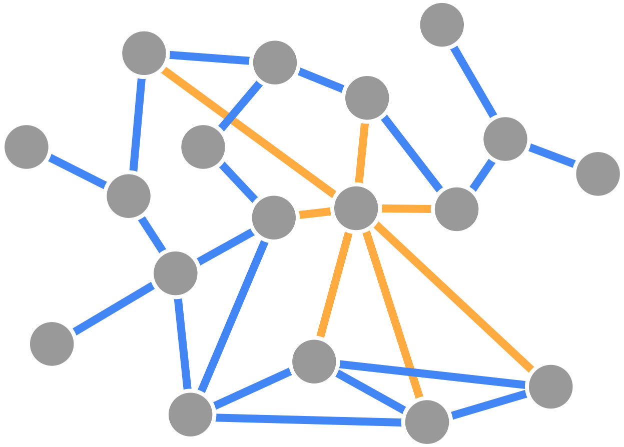
Latest Advances in Spectral Linear Algebra in **Network Science**

The scientific, computational, and mathematical study of real-life systems that can be represented as graphs.



Latest Advances in **Spectral Linear Algebra** in Network Science

Spectral graph theory. Eigenvector centrality. PageRank.



Latest Advances in Spectral Linear Algebra in Network Science

Modern topics:

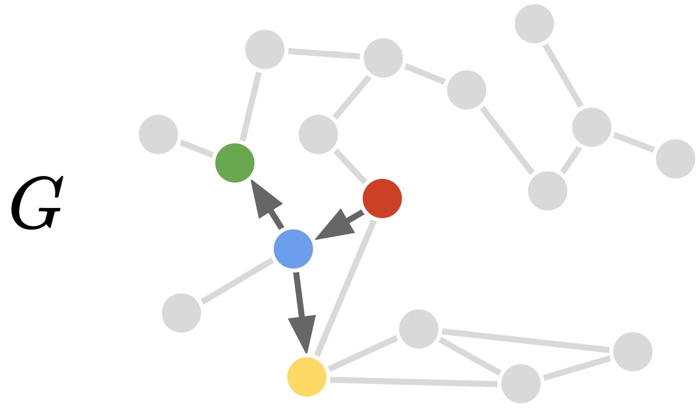
- graph signals
- graph symmetries
- generalizations
  - of graphs: simplicial complexes, hypergraphs
  - of matrices: tensors
- applications: synchronization, cybersecurity, epidemiology, etc
- “other” matrices e.g. non-backtracking matrix

Perturbation of Non-Backtracking Eigenvalues: Centrality and Diagonalizability

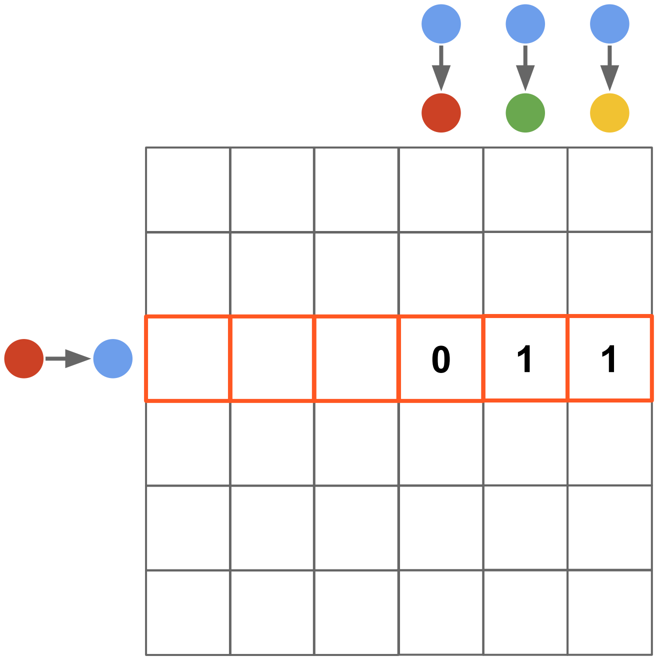
Leo Torres
with Kevin S. Chan, Hanghang Tong, Tina Eliassi-Rad



Non-backtracking Matrix



$$G = (V, E)$$
$$|E| = m$$



B

Non-backtracking eigenvalues

- length spectrum theory
  - Torres, et al. App. Net. Sci. 4.1 (2019): 41.
- community detection
  - Krzakala, et al. PNAS 110.52 (2013): 20935-20940.
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- epidemic thresholds (SIR, SIS)
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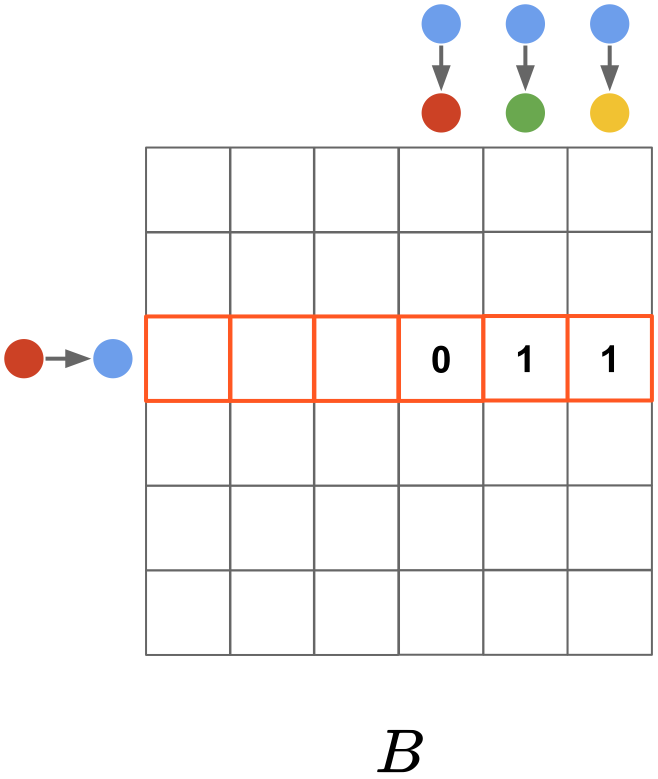

Non-backtracking eigenvalues

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$$\theta \approx 1/\lambda$$

What happens to the
leading eigenvalue of the
non-backtracking matrix
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What happens to the leading eigenvalue of the **non-backtracking** matrix when a node is removed from the graph?

$$\theta \approx 1/\lambda$$

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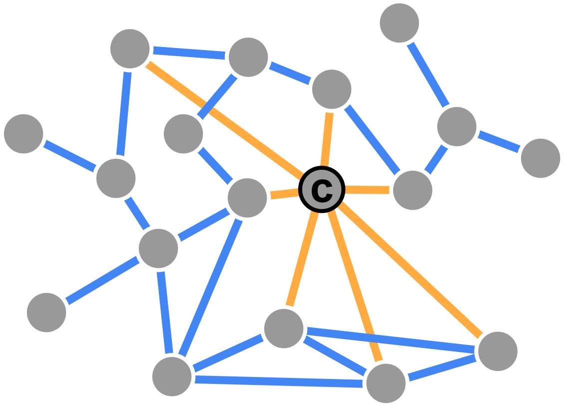
L. Torres, et al.

Nonbacktracking Eigenvalues under Node Removal: X-Centrality and Targeted Immunization.

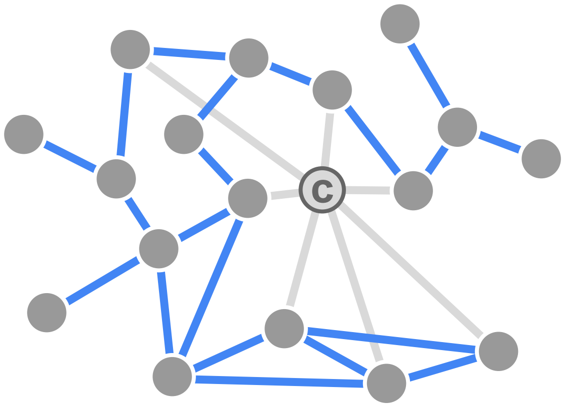
SIMODS, 3(2). 2021.

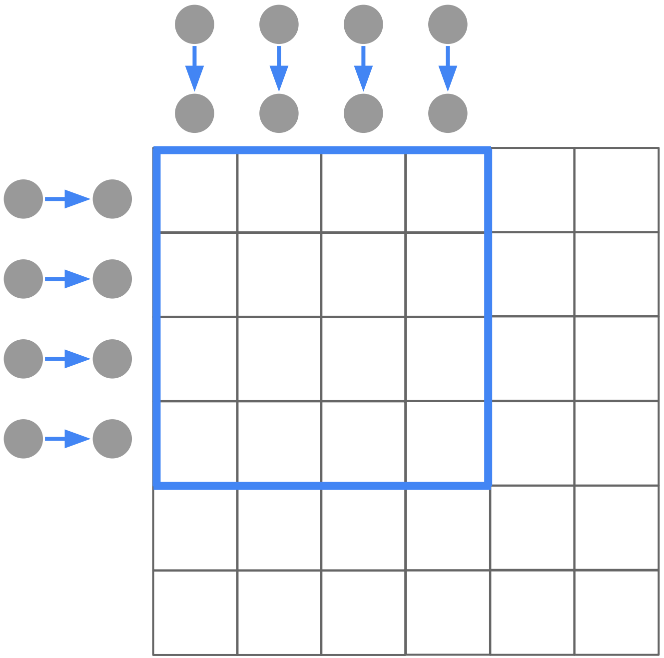
Setting



Block Matrix

G^c





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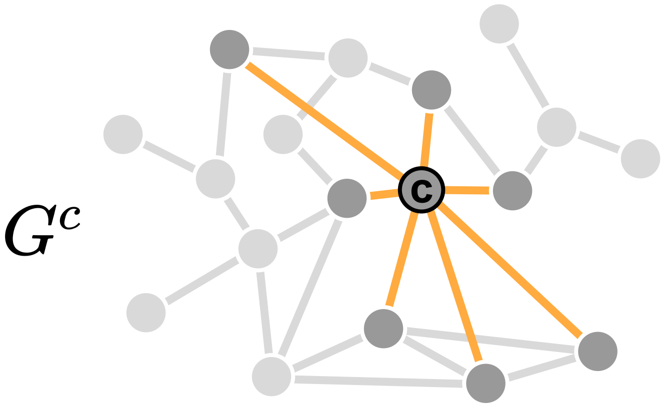
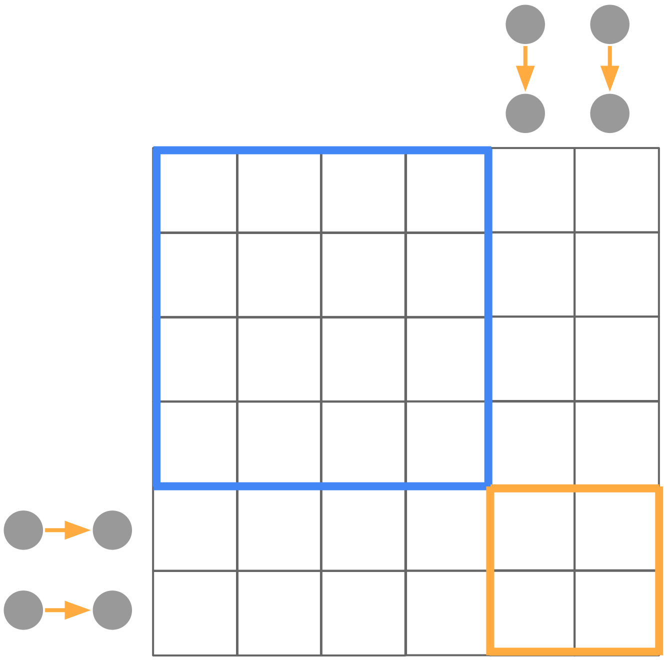
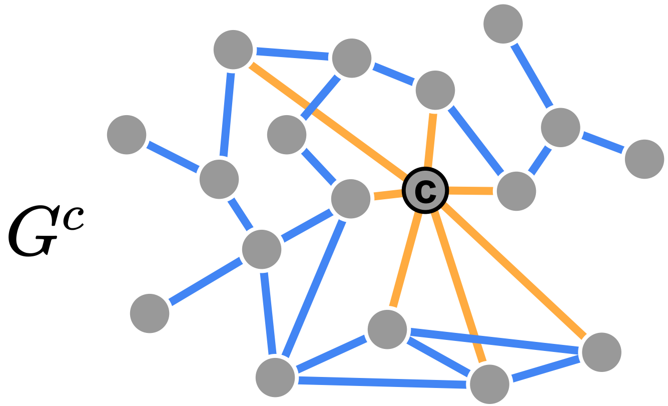
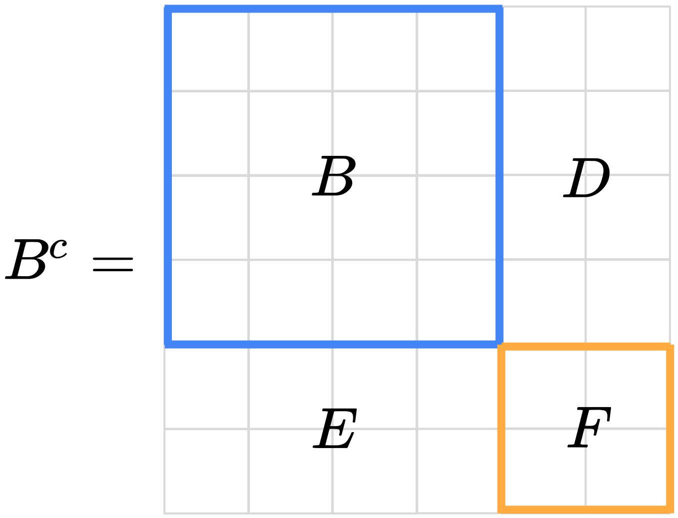




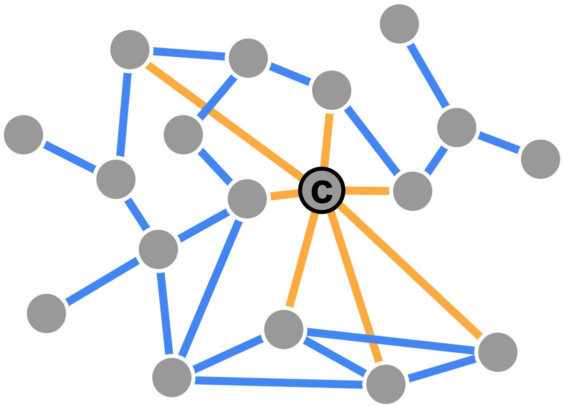
Block Matrix



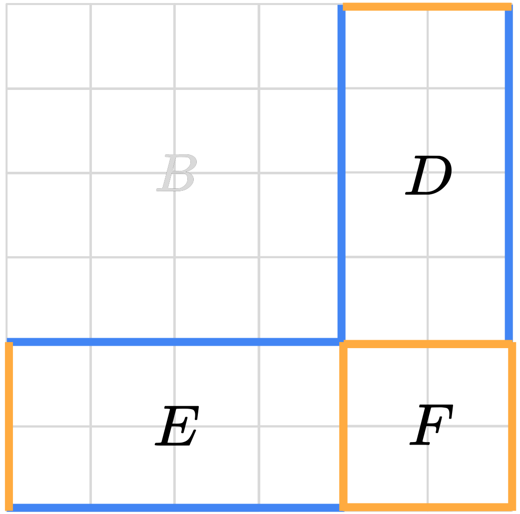


The X Matrix

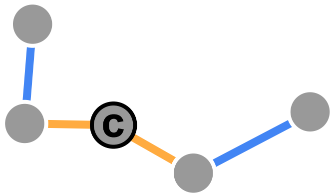
G^c



$B^c =$

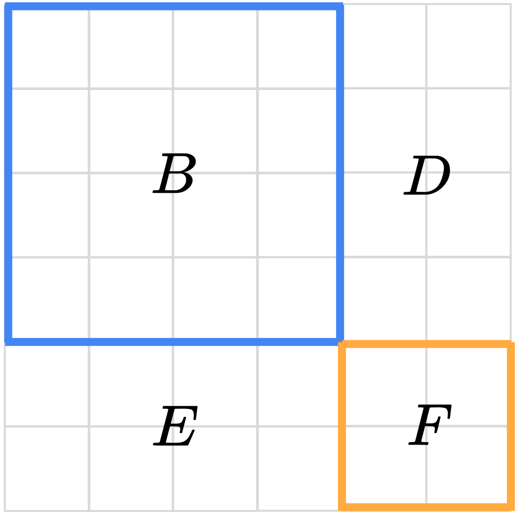


$X = DFE$



Solving for eigenvalues

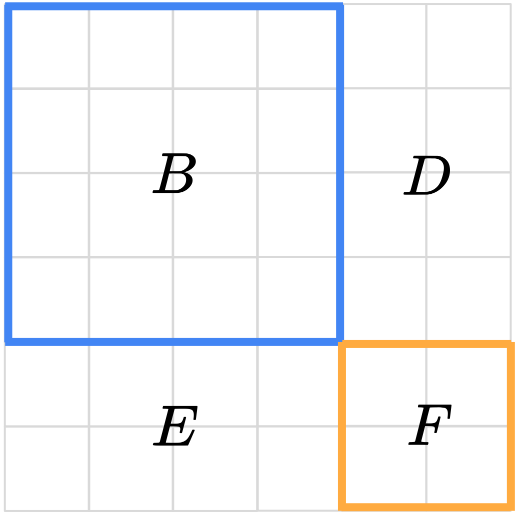
$B^c =$



$$\det\left(B^c - tI\right) = 0$$

Solving for eigenvalues

$B^c =$



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• • •

L. Torres, et al.

Nonbacktracking Eigenvalues under Node Removal: X-Centrality and Targeted Immunization.

SIMODS, 3(2). 2021.

Solving for eigenvalues

$$B^c = \begin{pmatrix} B & D \\ E & F \end{pmatrix}$$

$$X = DFE$$

u_1, v_1 = **right and left eigenvectors**

$$\det\left(B^c - tI\right) = 0$$

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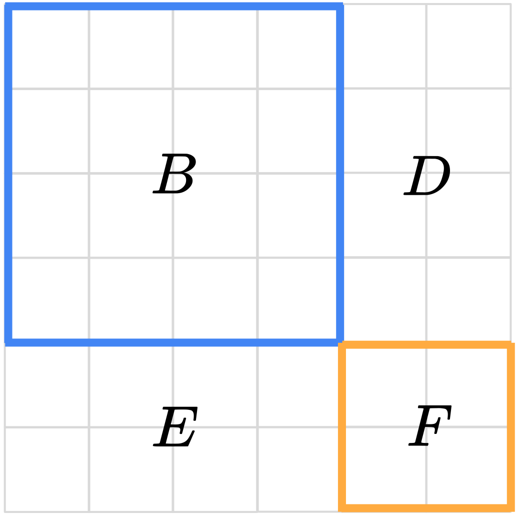
$$t^2(t - \lambda_1) + v_1^T X u_1 = 0$$

Solving for eigenvalues

$$B^c =$$



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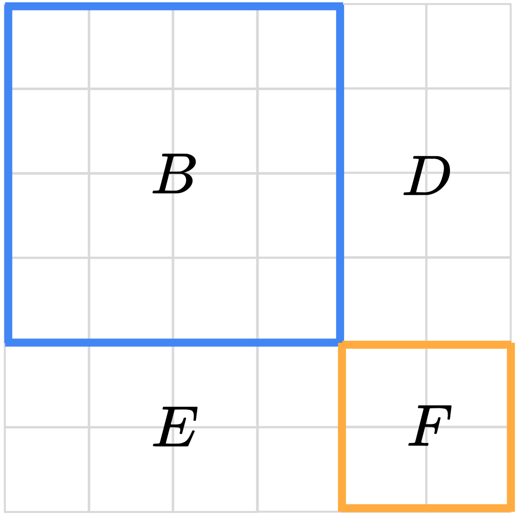
$$t^2\left(t - \lambda_1\right) + \boxed{v_1^T X u_1} = 0$$

XNB Centrality

$$B^c =$$



$$X = DFE$$

$$\det\left(B^c - tI\right) = 0$$

• • •

X-non-backtracking centrality

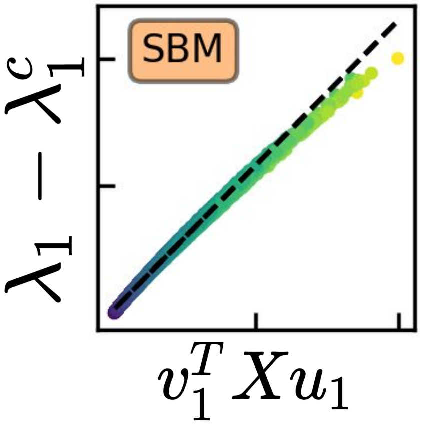
$$t^2(t - \lambda_1) + \boxed{v_1^T X u_1} = 0$$

u_1, v_1 = **right and left eigenvectors**

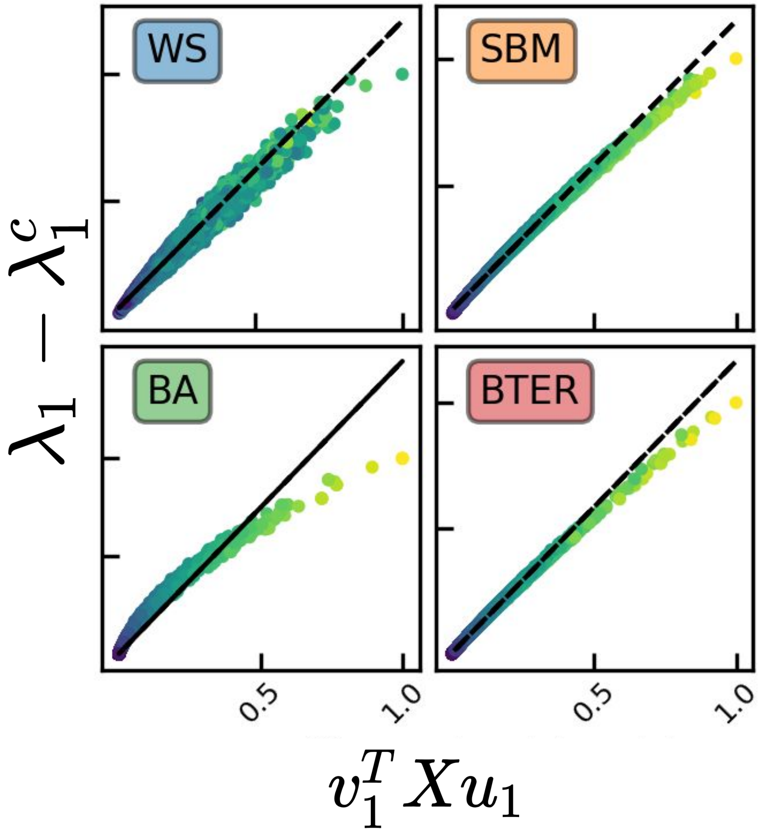
What happens to the
leading eigenvalue of the
non-backtracking matrix
when a node is removed
from the graph?

It **decreases** by a quantity that is **correlated** to $v_1^T X u_1$

XNB and the true change in eigenvalue



XNB and the true change in eigenvalue



Now what?

Deriving this expression

- Many ways of deriving this...

$$t^2(t - \lambda_1) + v_1^T X u_1 = 0$$

Deriving this expression

- Many ways of deriving this...
- The cleanest way assumes the matrix is diagonalizable.

$$t^2(t - \lambda_1) + v_1^T X u_1 = 0$$

Deriving this expression

- Many ways of deriving this...
- The cleanest way assumes the matrix is diagonalizable.
- **When is the non-backtracking matrix diagonalizable?**

$$t^2(t-\lambda_1)+v_1^T X u_1 = 0$$

Deriving this expression

- When is the non-backtracking matrix diagonalizable?

L. Torres

Non-backtracking Spectrum: Unitary Eigenvalues and Diagonalizability.

arXiv preprint arXiv:2007.13611 (2020).

Deriving this expression

- When is the non-backtracking matrix diagonalizable?
  - What is the multiplicity of the eigenvalues with unit magnitude?

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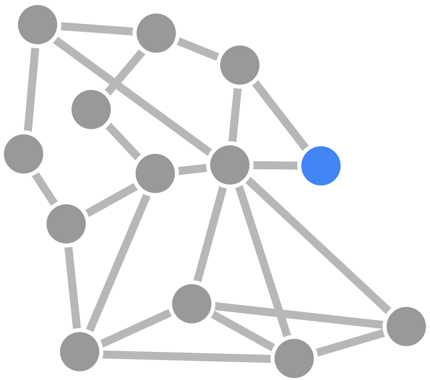
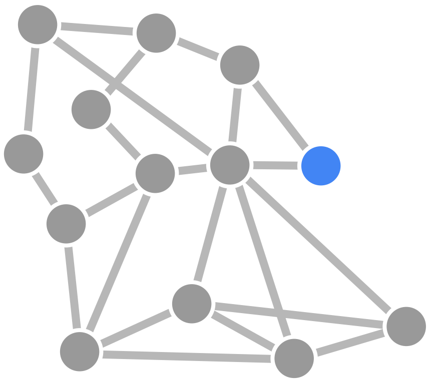
What is the multiplicity of the eigenvalues with unit magnitude?

G

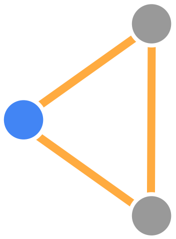


What is the multiplicity of the eigenvalues with unit magnitude?

G



C_3



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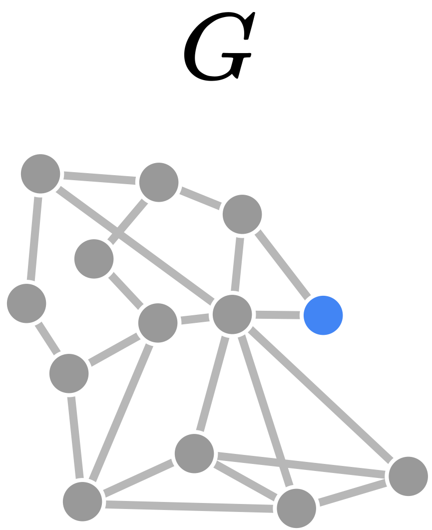

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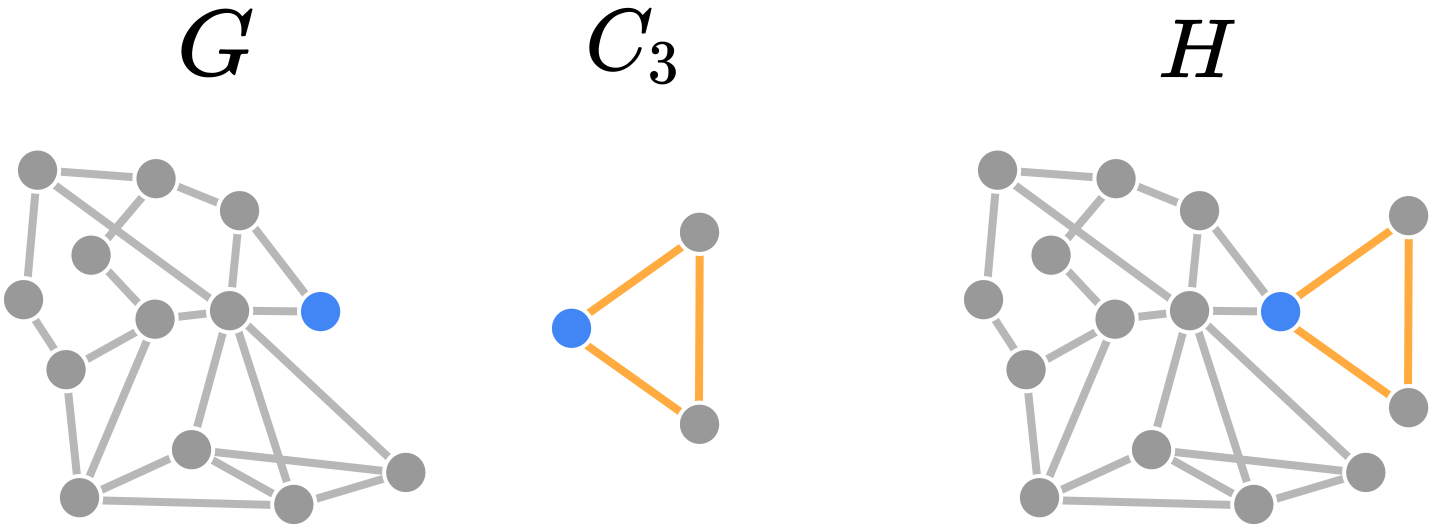
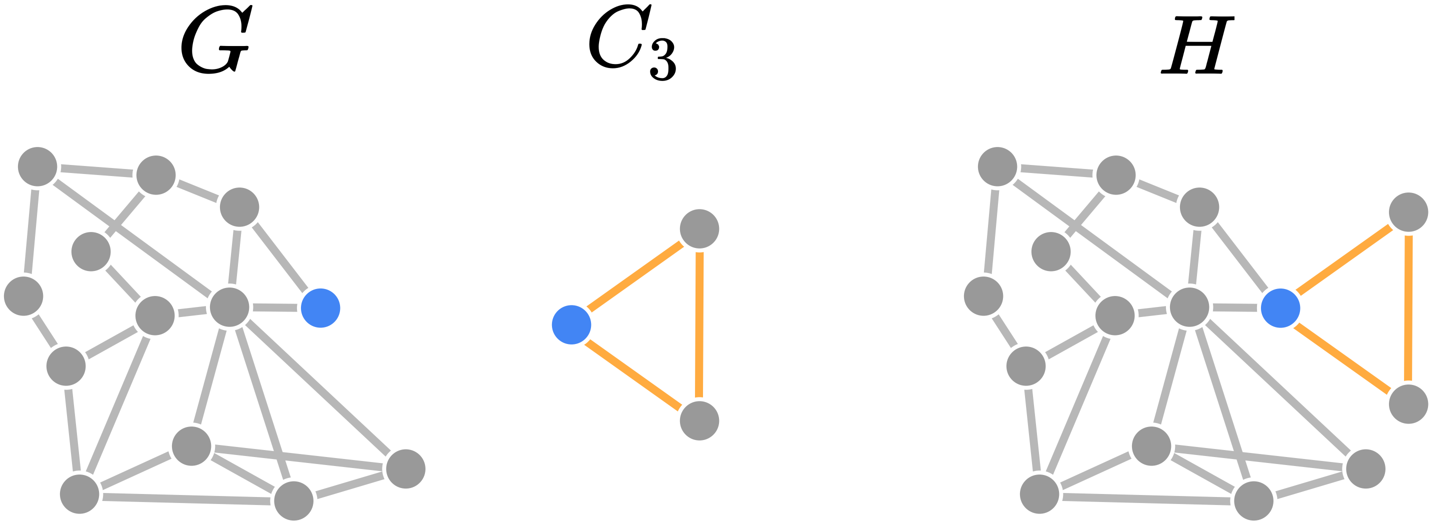


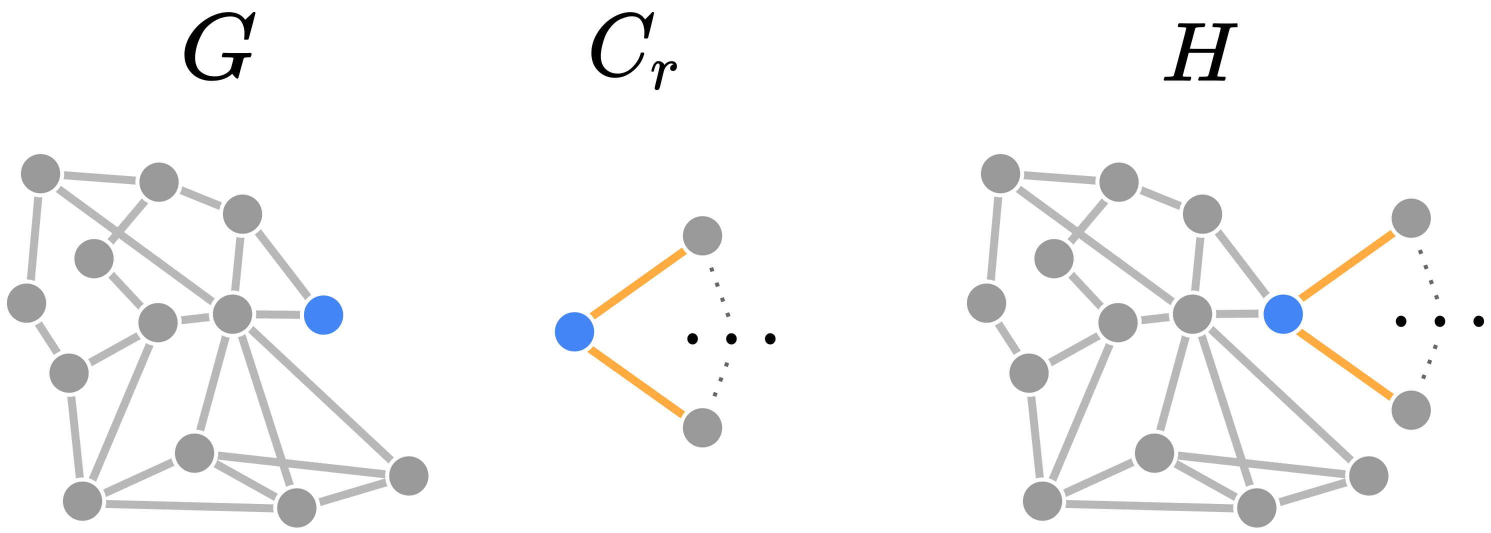
$$m_H$$

Cubic roots of unity



$$m_G + 1 = m_H$$

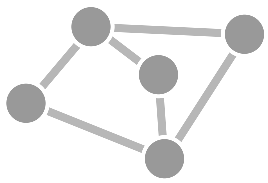
Arbitrary roots of unity



$$m_G + 1 = m_H$$

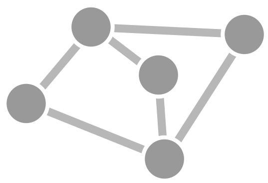
Subdivisions

S

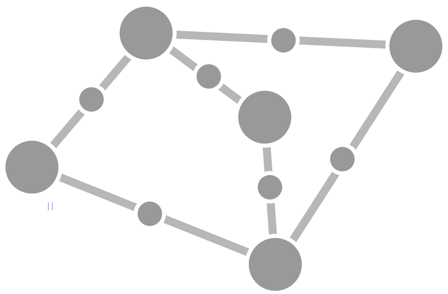


Subdivisions

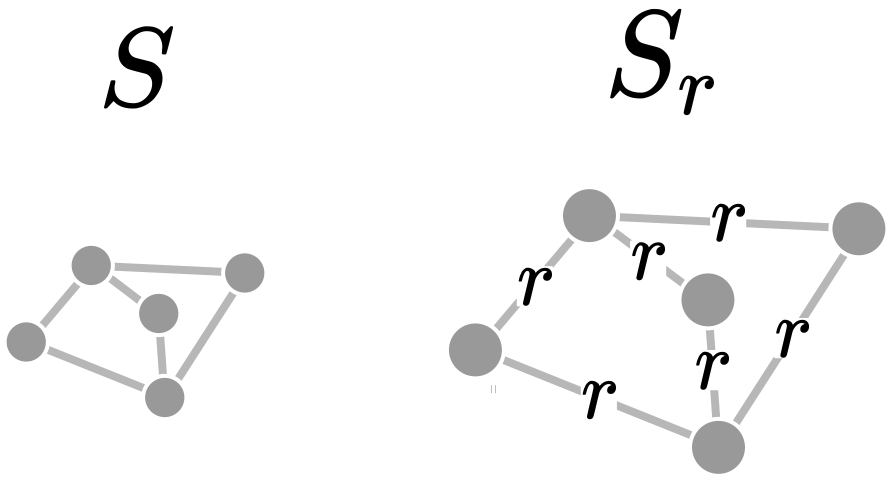
S



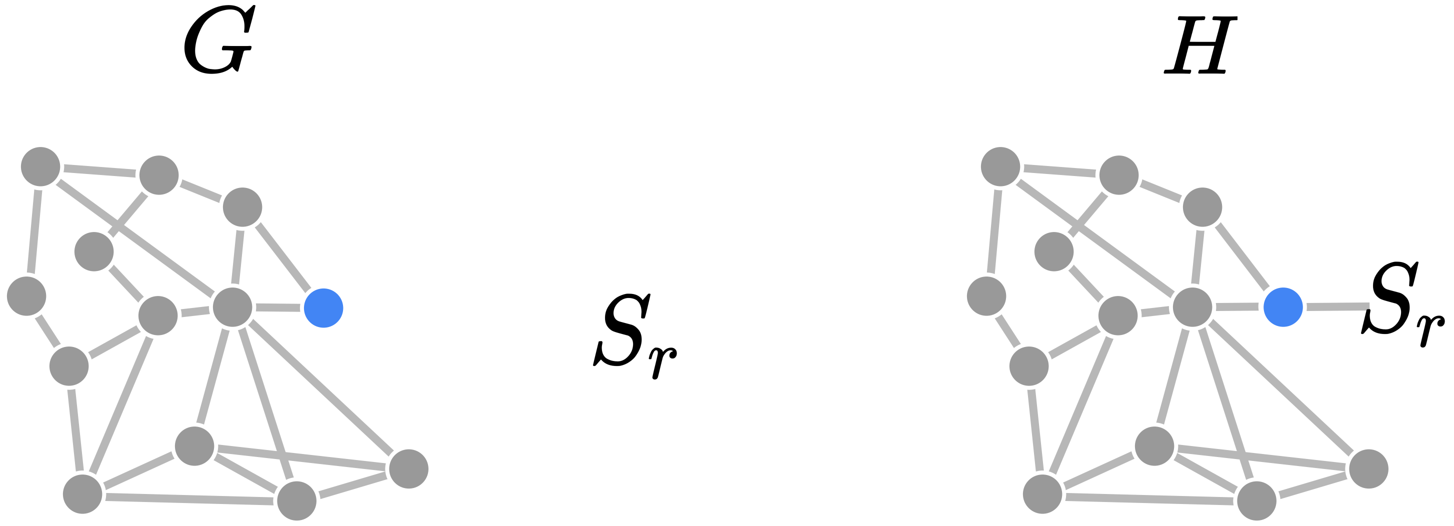
S_2



Subdivisions



Arbitrary subdivisions



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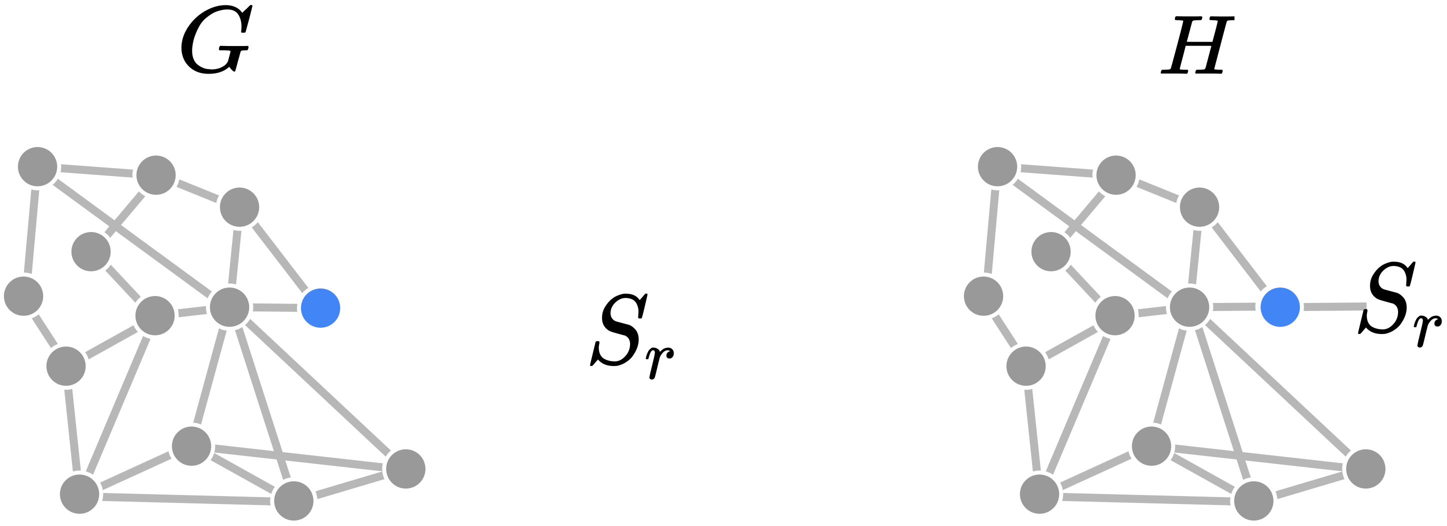
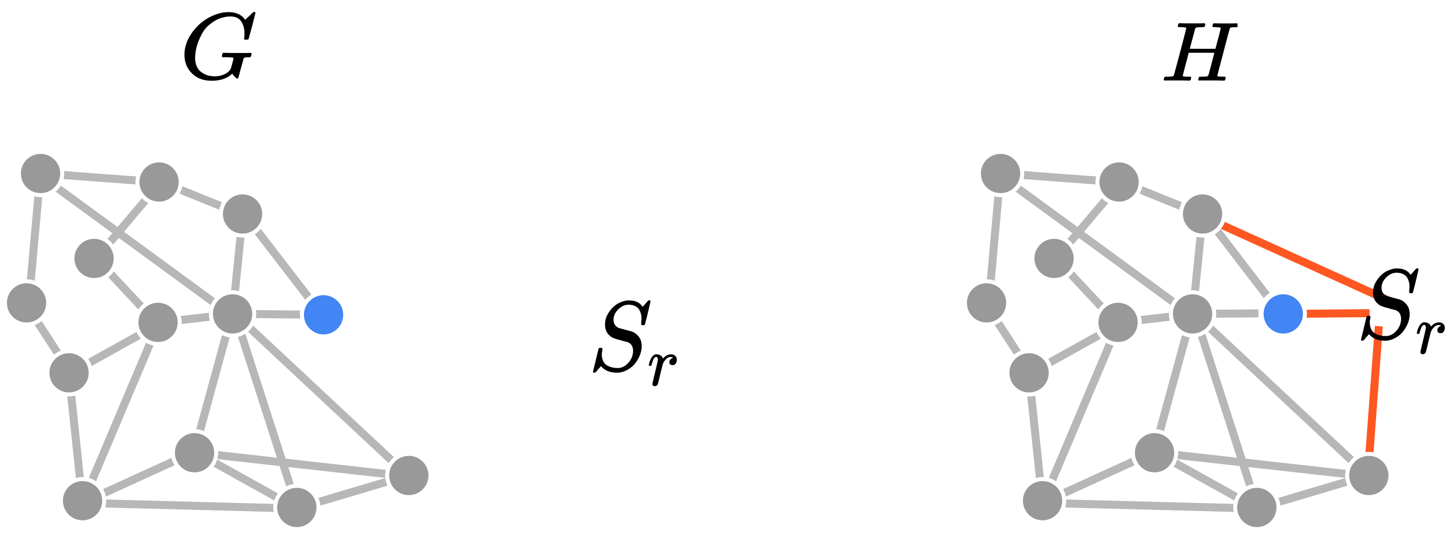


$$m_G + m_{S_r} = m_H$$

Arbitrary gluings



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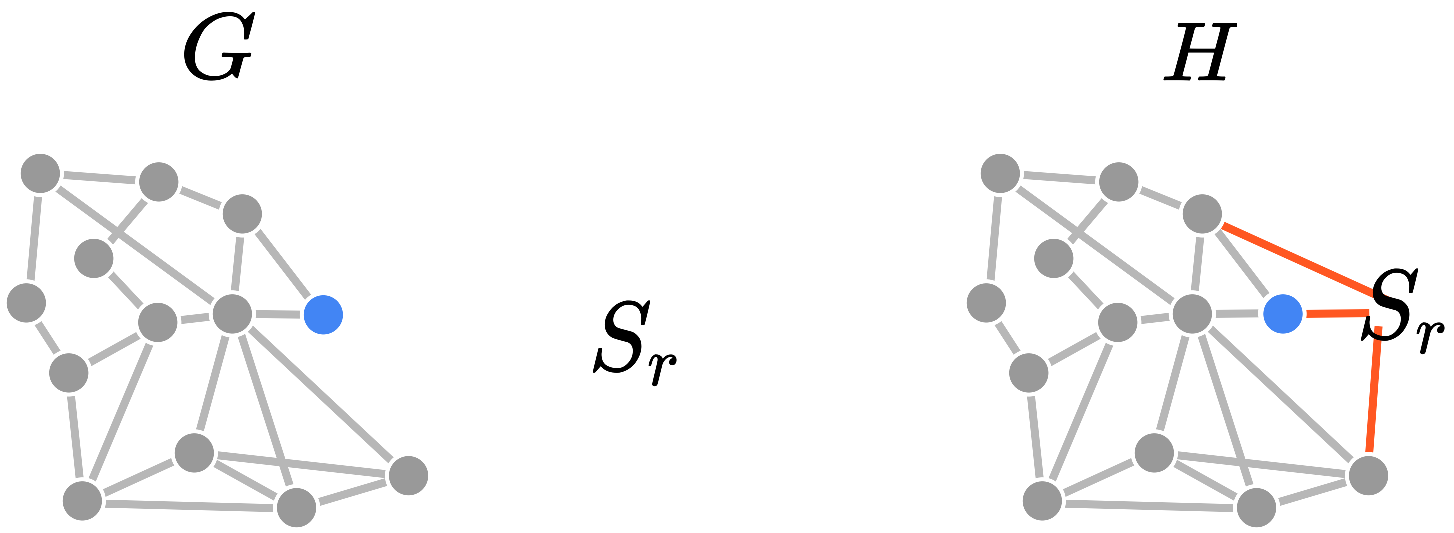

Theorem. Gluing a graph subdivision is the only way that unitary eigenvalues appear in the spectrum.

Gracias!

L. Torres, et al.

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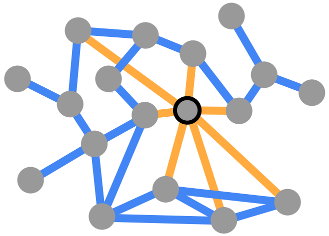
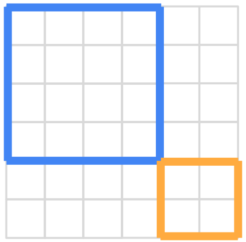
SIMODS, 3(2). 2021.





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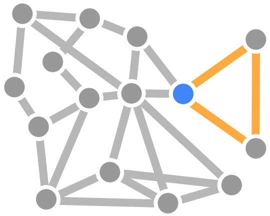
arXiv preprint arXiv:2007.13611 (2020 - **old version!**).



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