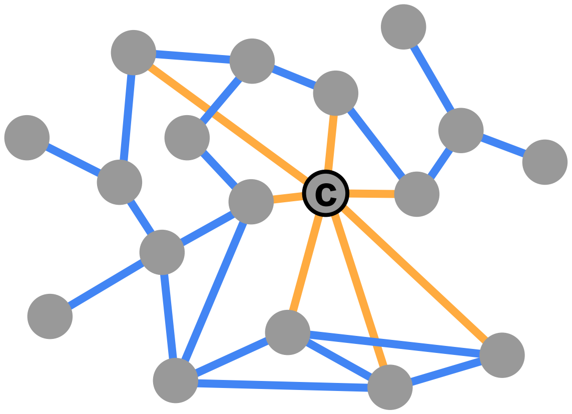
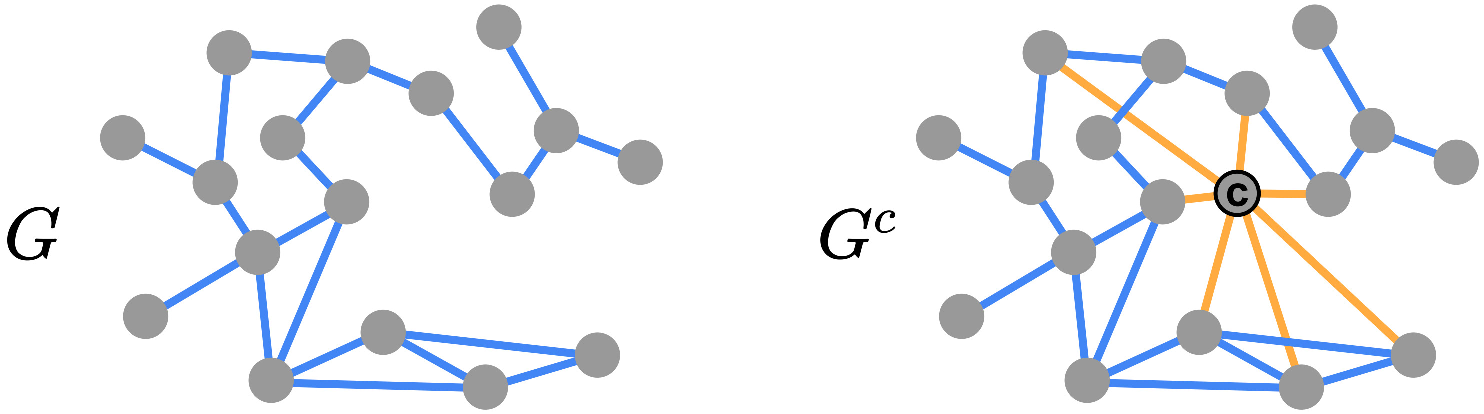
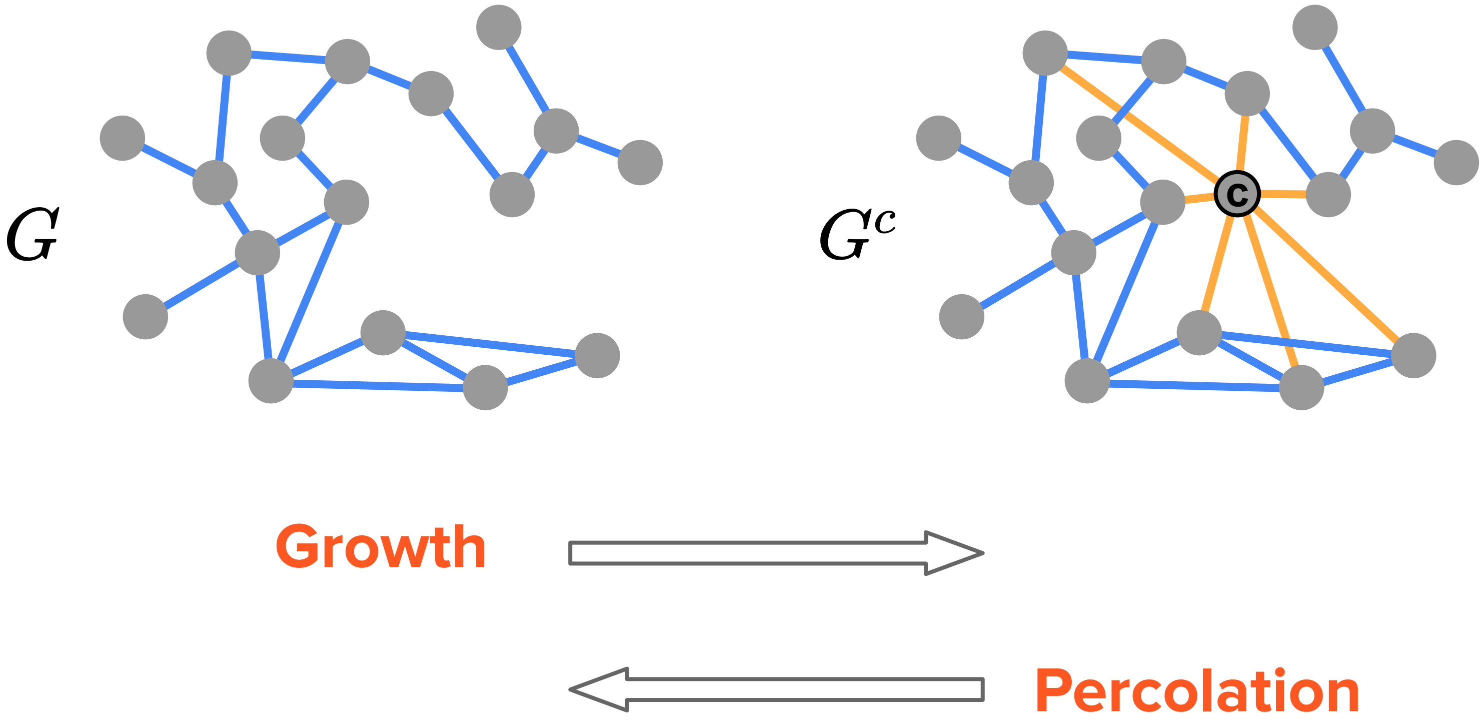
The largest non-backtracking eigenvalue under node removal

Leo

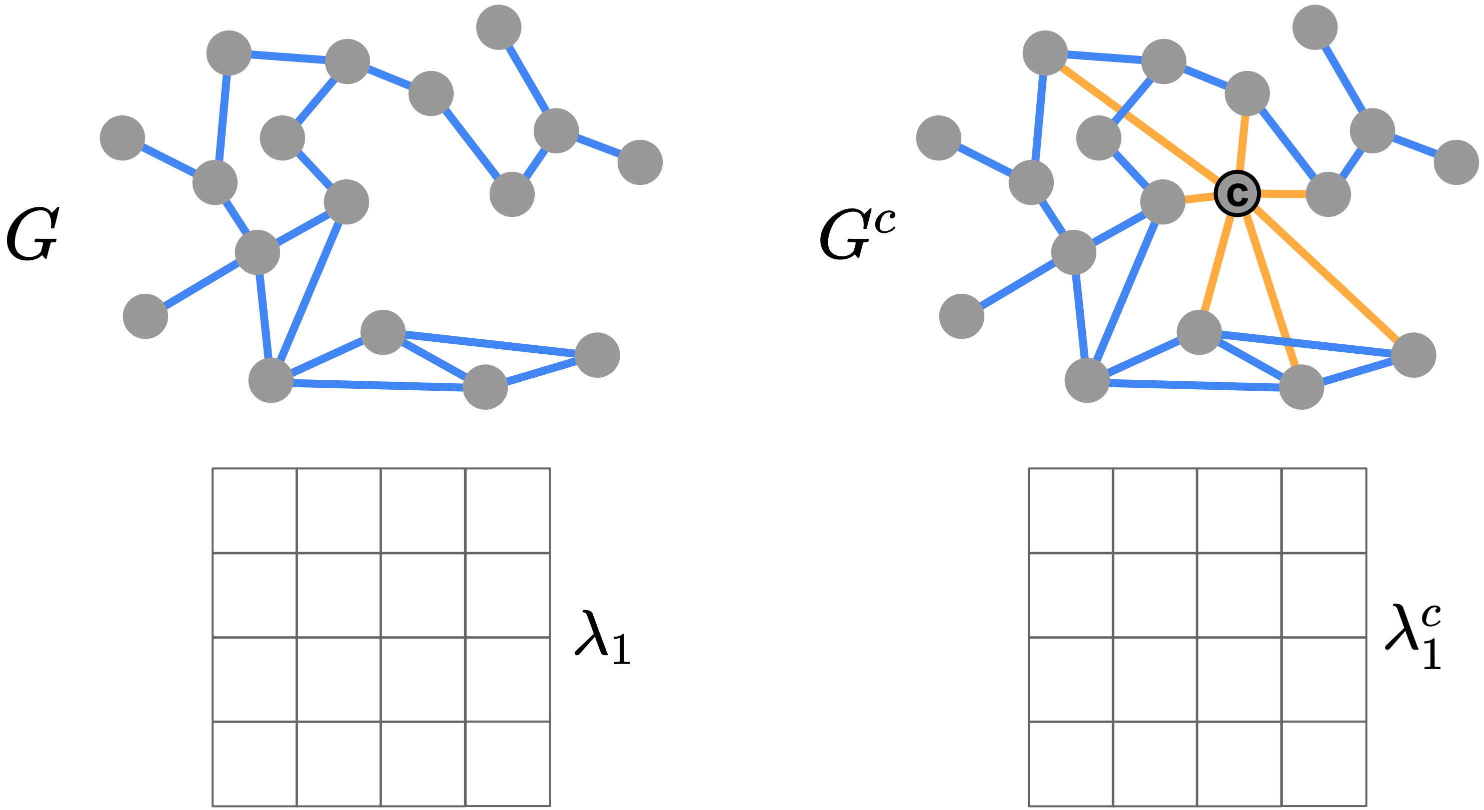
G^c




G
G^c
c

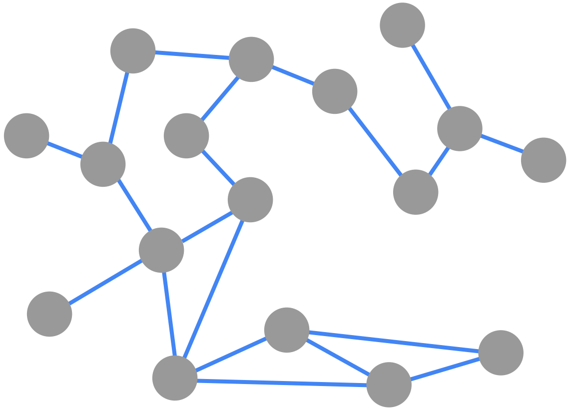


G
G^c
c
Growth
Percolation



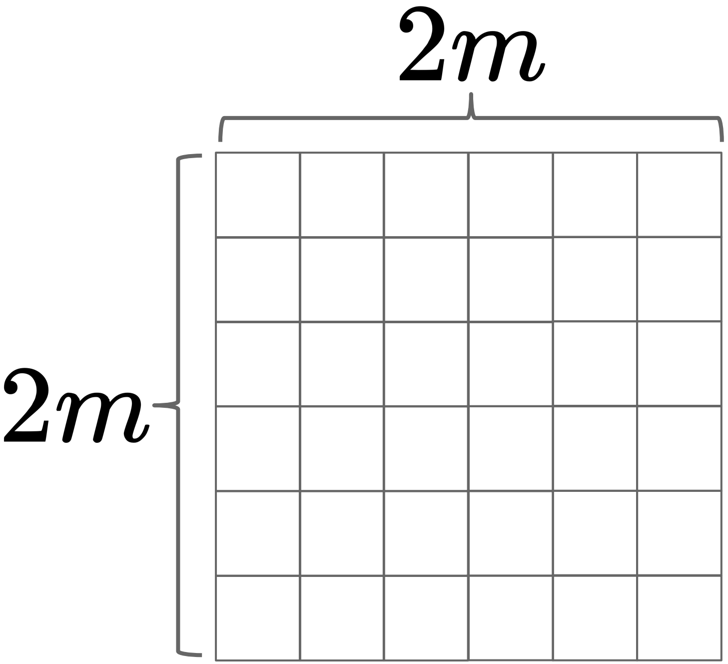
G
G^c
c
λ_1
λ_1^c

G

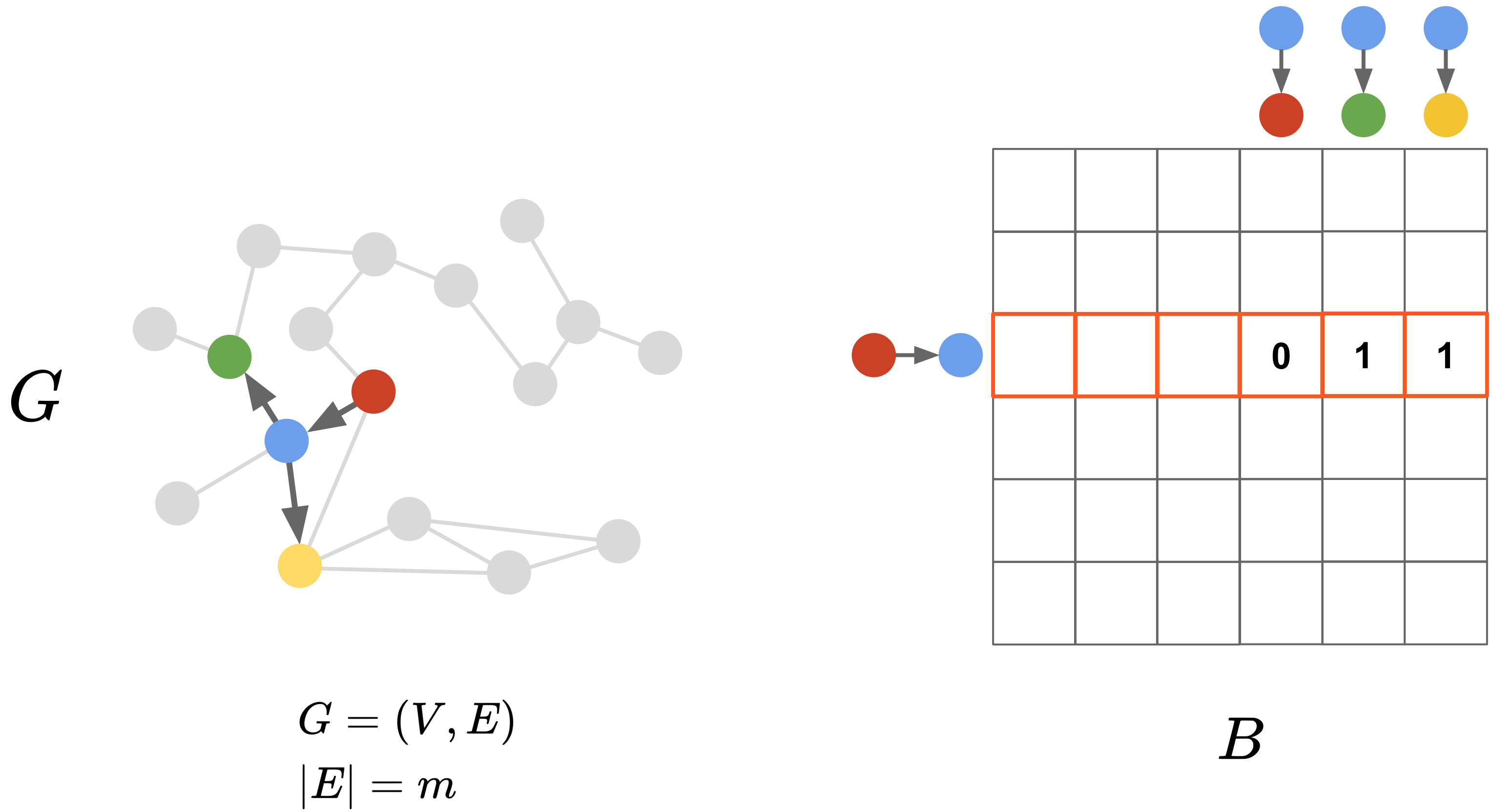


$G = (V, E)$

$|E| = m$

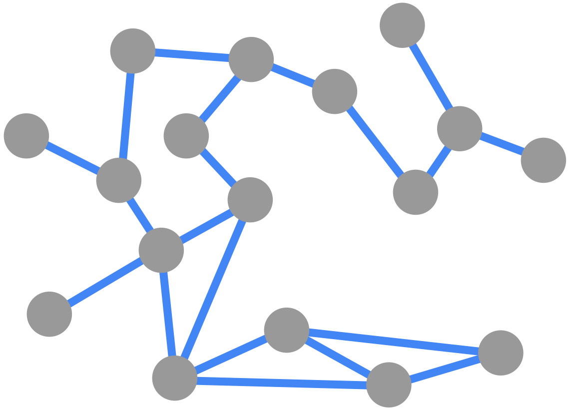


B

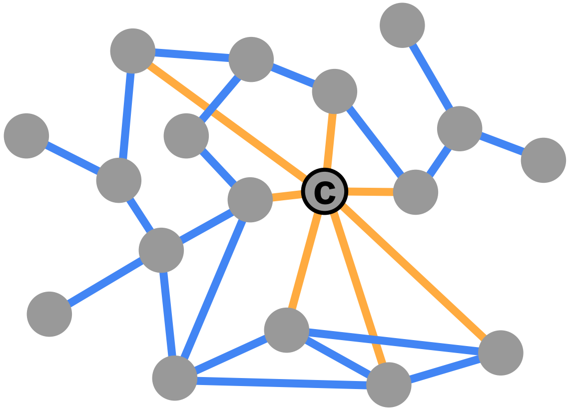

G
0
1
1
$G = (V, E)$
$|E| = m$
B

The largest non-backtracking eigenvalue under node removal

G



G^c



B, λ_1

B^c, λ_1^c

Outline

1. Problem definition
2. Motivation
3. Block matrix
4. Finding a root
5. Application: immunization
6. Application: detecting dense subgraphs

Motivation

Mathematics

Network Science

Motivation

Mathematics

1. Non-backtracking matrix is not symmetric and not normal.
2. No eigenvalue interlacing.

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Motivation

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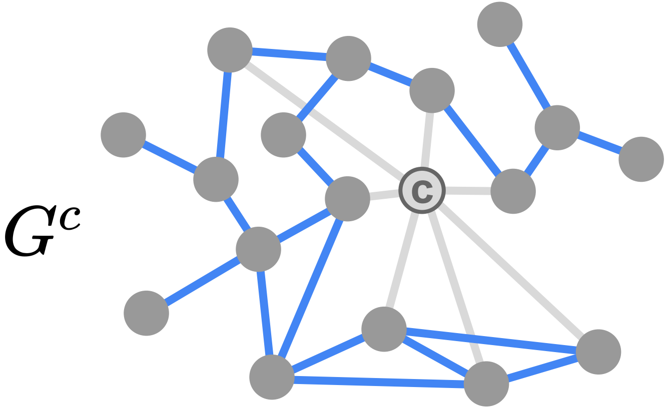
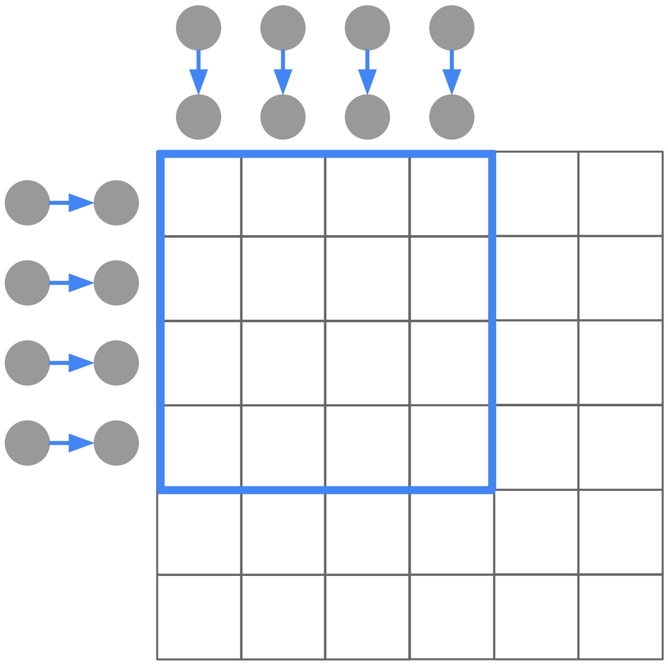
Network Science

1. Growth, percolation, dynamics of.
2. The spread of certain epidemics is described by λ_1, (Shrestha, Scarpino, Moore, 2015).

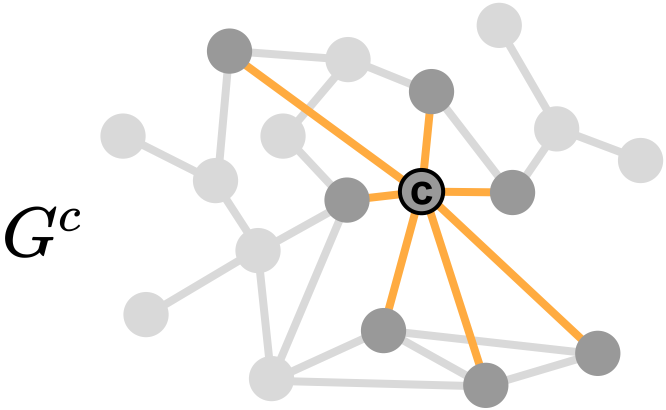
Application: which node to remove to decrease λ_1 the most?

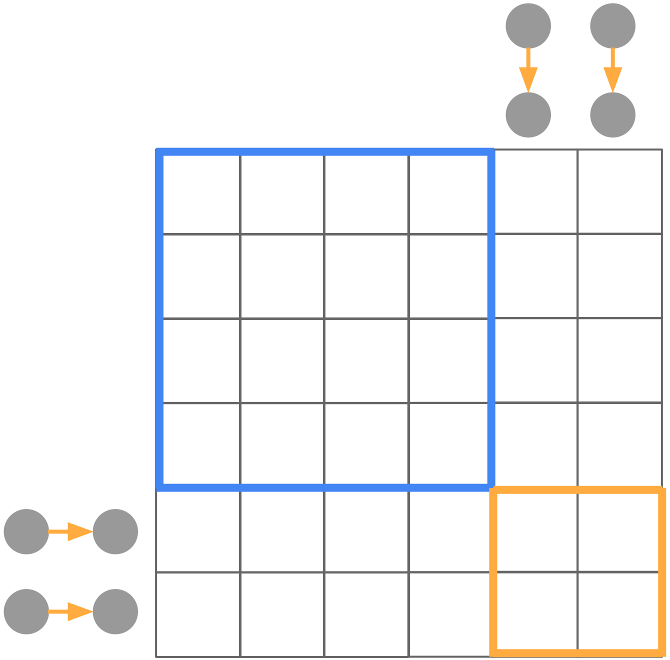
Block Matrix



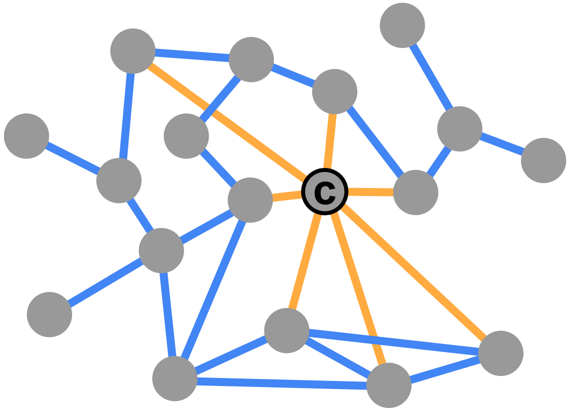


Block Matrix





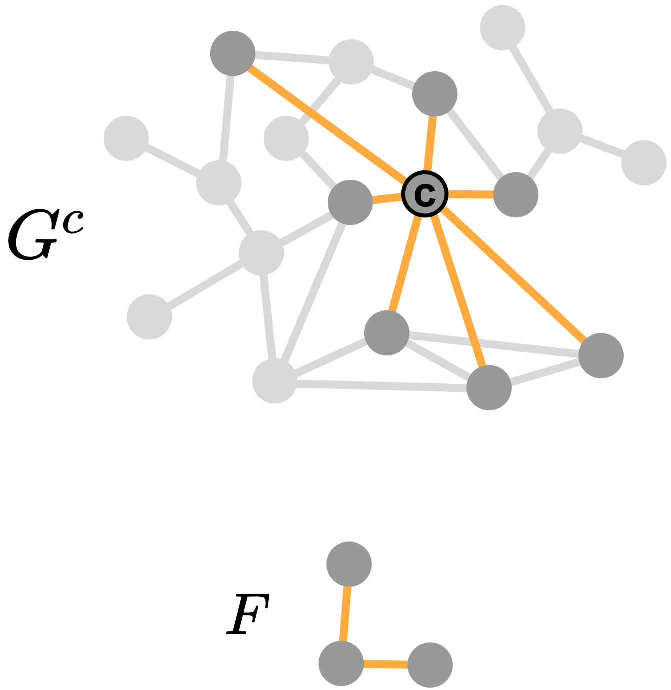
Block Matrix

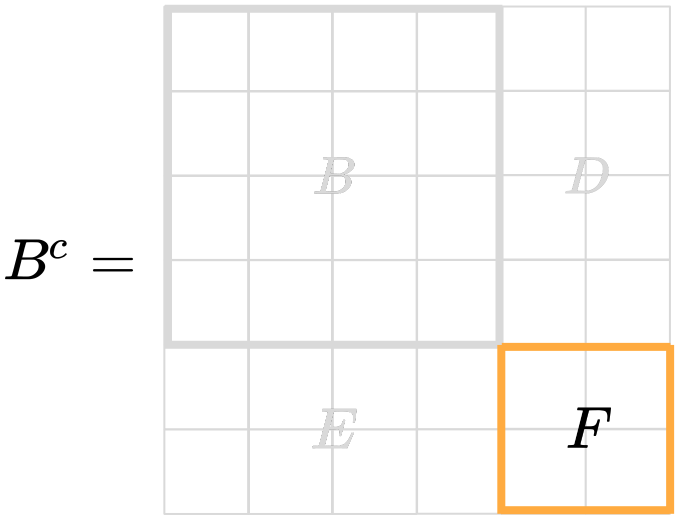


G^c

$$B^c = \begin{bmatrix} B & D \\ E & F \end{bmatrix}$$

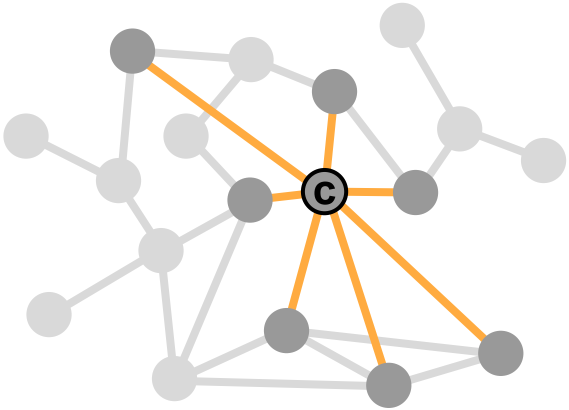
Block Matrix



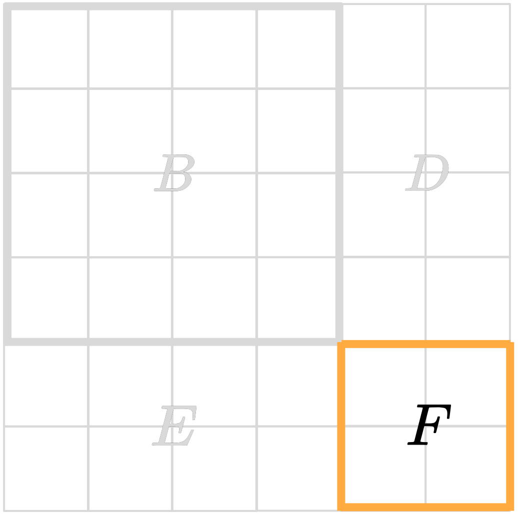


Block Matrix

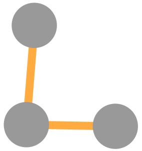
G^c



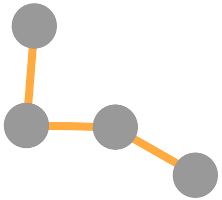
$B^c =$



F



F^2



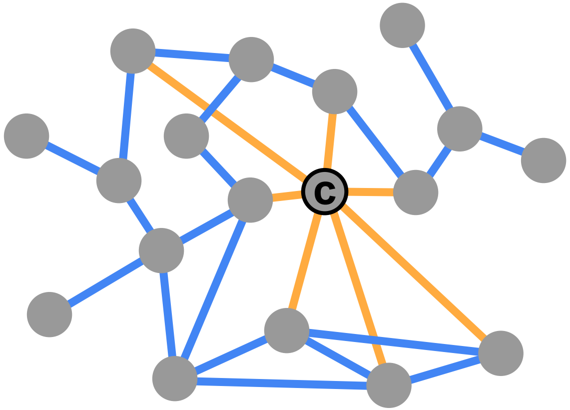
$F^2 = 0$

Block Matrix

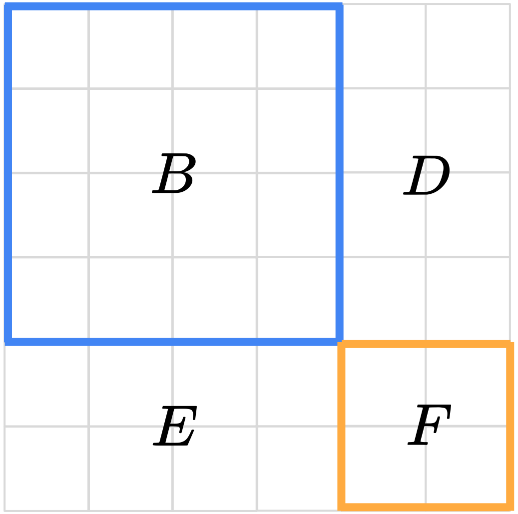
Score card

$F^2 = 0$

G^c



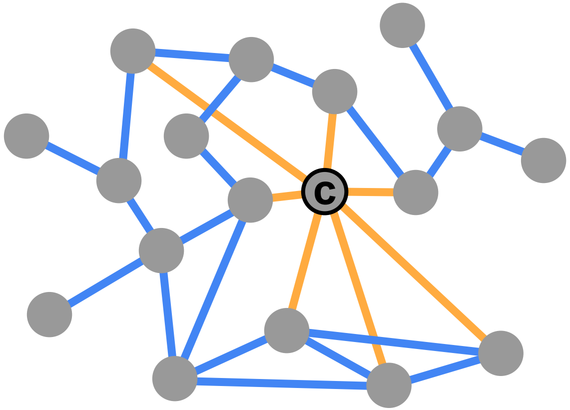
$B^c =$

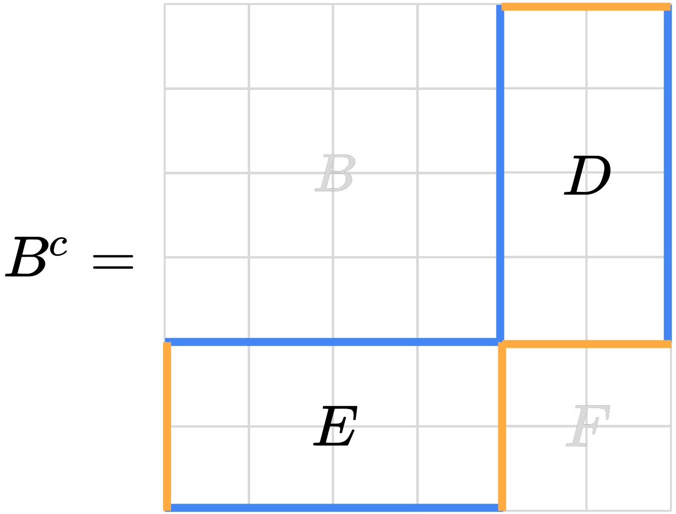


$F^2 = 0$

Block Matrix

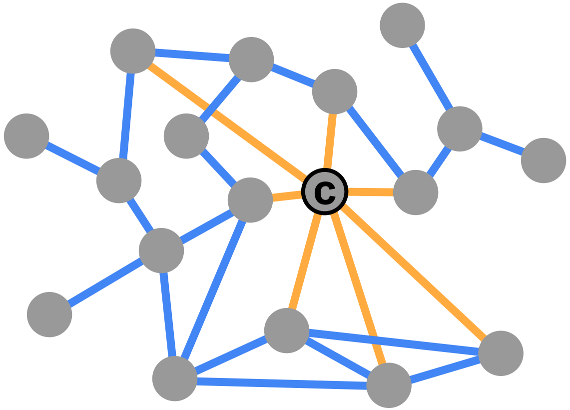
G^c



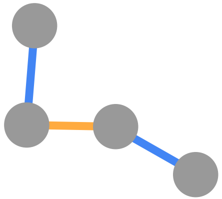


Block Matrix

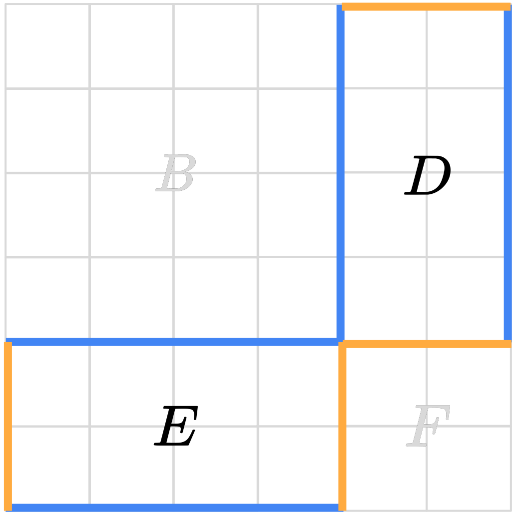
G^c



DE



$B^c =$

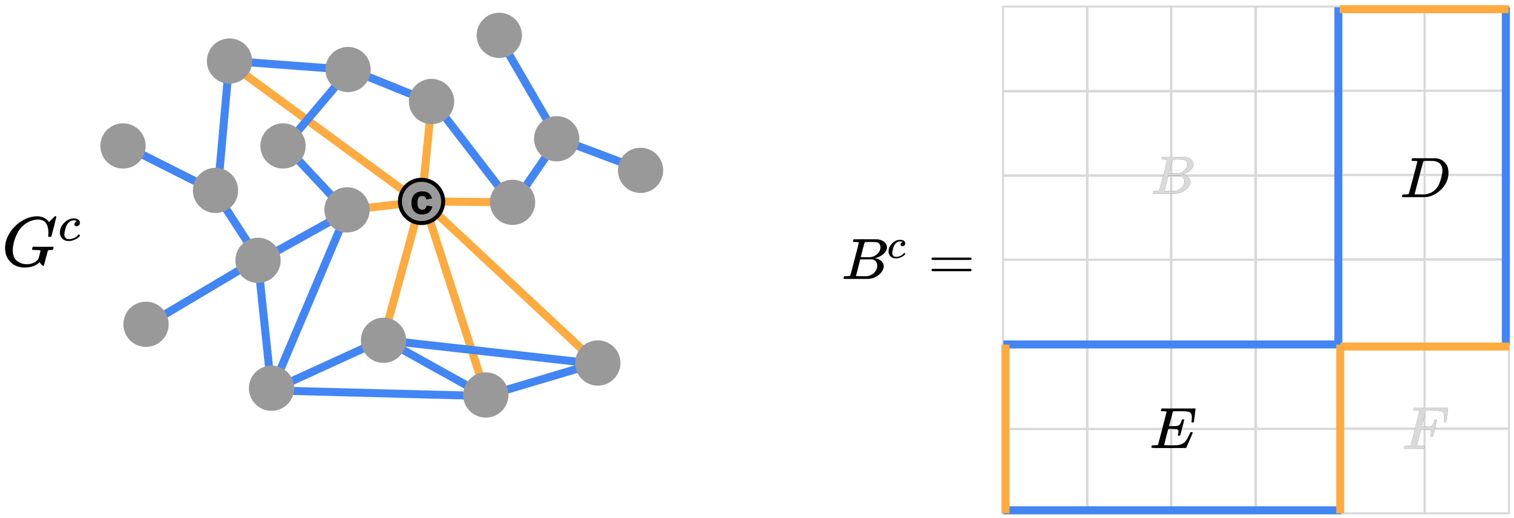


$DE = 0$

Block Matrix

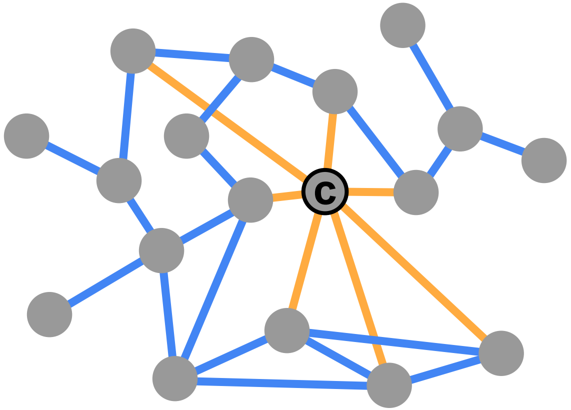
Score card

$F^2 = 0$
$DE = 0$

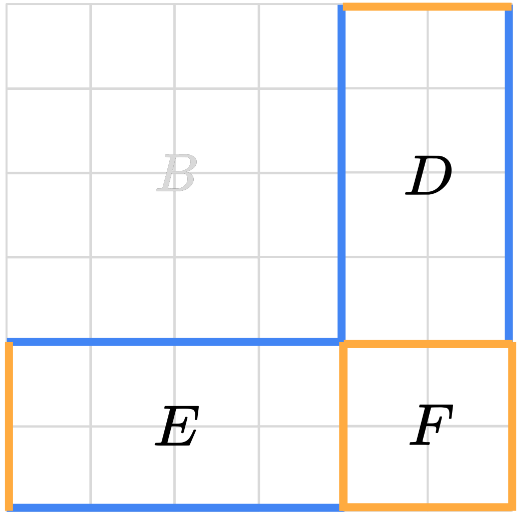


Block Matrix

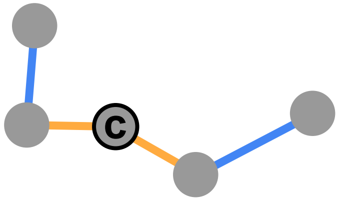
G^c



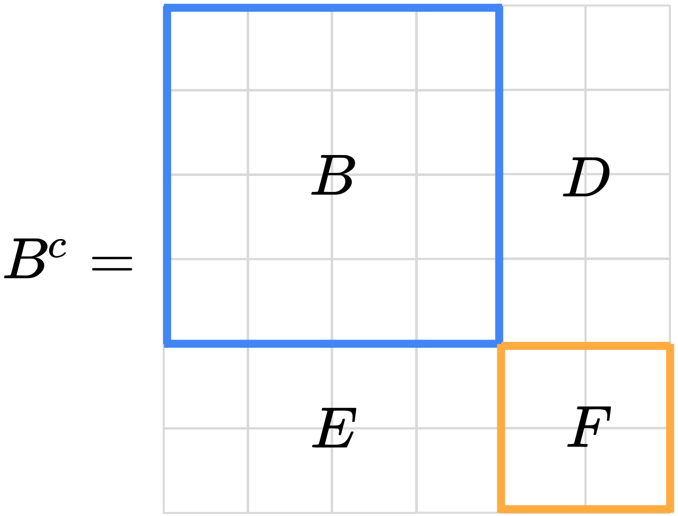
$B^c =$



$X = DFE$



Solving for eigenvalues

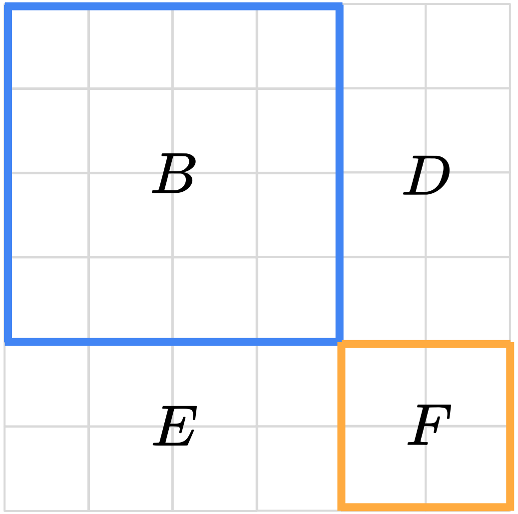


Solve:

$$\det\left(B^c - tI\right) = 0$$

Solving for eigenvalues

$B^c =$



Score card

$F^2 = 0$

$DE = 0$

$X = DFE$!!!

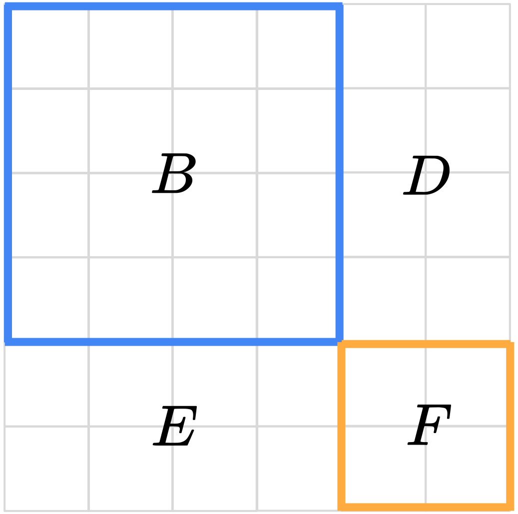
Solve:

$$\det\left(B^c - tI\right) = 0$$

$$\det\left(B^c - tI\right) = \det\left(F - tI\right)\det\left(B - tI - D(F - tI)^{-1}E\right)$$

Solving for eigenvalues

$B^c =$



Score card

$$F^2 = 0$$
$$DE = 0$$
$$X = DFE!!!$$

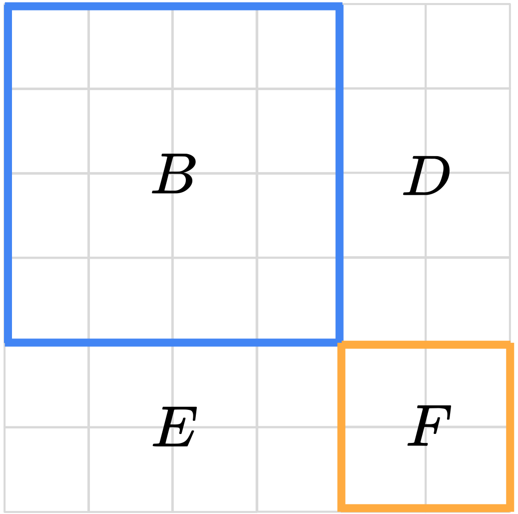
Solve:

$$\det\left(B^c - tI\right) = 0$$

$$\begin{aligned}\det\left(B^c - tI\right) &= \det\left(F - tI\right)\det\left(B - tI - D(F - tI)^{-1}E\right)\\ &= t^{2d}\det\left(B - tI + \frac{X}{t^2}\right)\end{aligned}$$

Solving for eigenvalues

$B^c =$



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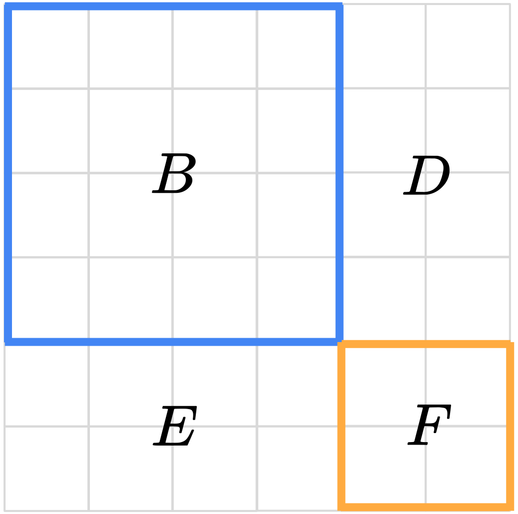
$$Y = (B - tI)^{-1}$$

Solving for eigenvalues

$$B^c =$$



Solve:

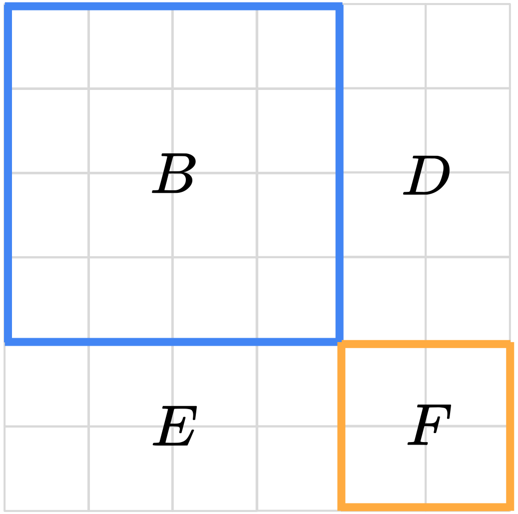
$$\det\left(B^c - tI\right) = 0$$

$$\det\left(B^c - tI\right) = t^{2d}\det\left(B - tI\right)\det\left(I + \frac{YX}{t^2}\right)$$

$$Y = \left(B - tI\right)^{-1}$$

Solving for eigenvalues

$B^c =$



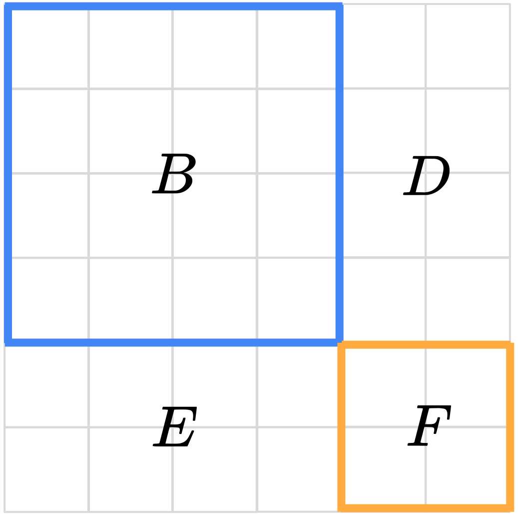
Solve:

$$\det\left(I + \frac{YX}{t^2}\right) = 0$$

$$X = DFE$$
$$Y = (B - tI)^{-1}$$

Solving for eigenvalues

$B^c =$



$$\det\left(I + \frac{YX}{t^2}\right) = 1 + \frac{1}{t^2} Tr\,(YX) + \ldots$$

Solve:

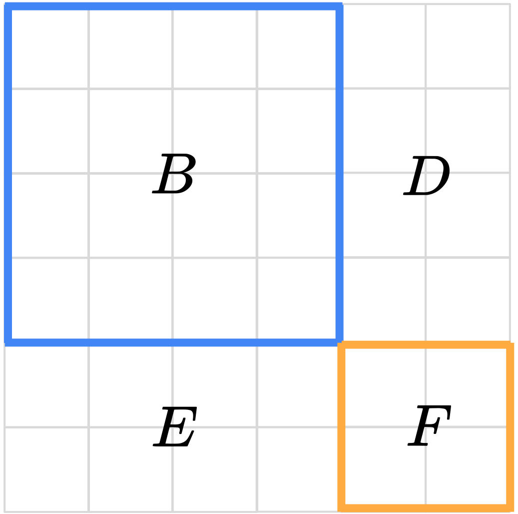
$$\det\left(I + \frac{YX}{t^2}\right) = 0$$

$$X = DFE$$

$$Y = (B - tI)^{-1}$$

Solving for eigenvalues

$B^c =$



$$\det\left(I + \frac{YX}{t^2}\right) = 1 + \frac{1}{t^2}\sum_i \frac{v_i^T X u_i}{t - \lambda_i} + \ldots$$

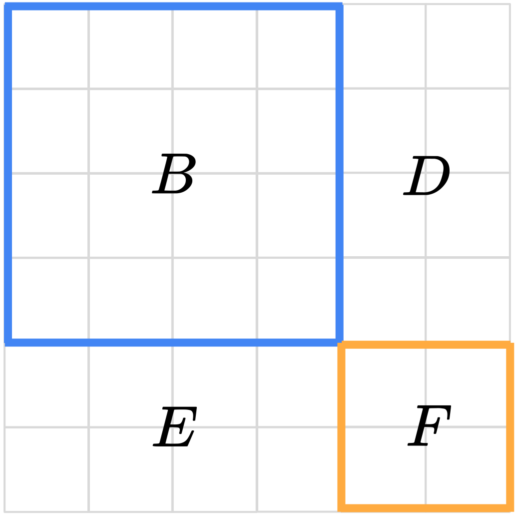
Solve:

$$\det\left(I + \frac{YX}{t^2}\right) = 0$$

$$X = DFE$$
$$Y = (B - tI)^{-1}$$

Solving for eigenvalues

$B^c =$



Solve:

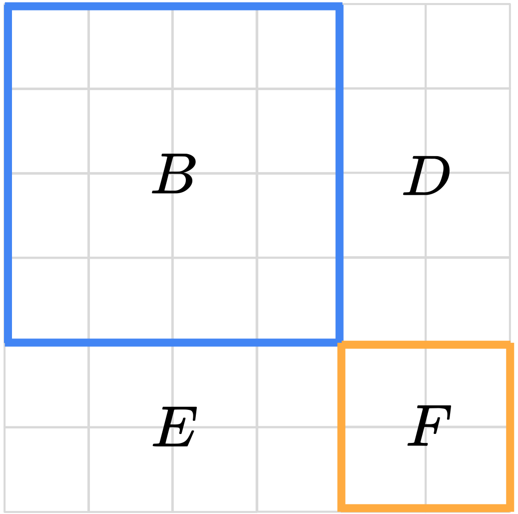
$$\det\left(I + \frac{YX}{t^2}\right) = 0$$

$$X = DFE$$
$$Y = (B - tI)^{-1}$$

$$\det\left(I + \frac{YX}{t^2}\right) = 1 + \frac{1}{t^2}\frac{v_1^T X u_1}{t-\lambda_1} + \dots$$

Solving for eigenvalues

$B^c =$



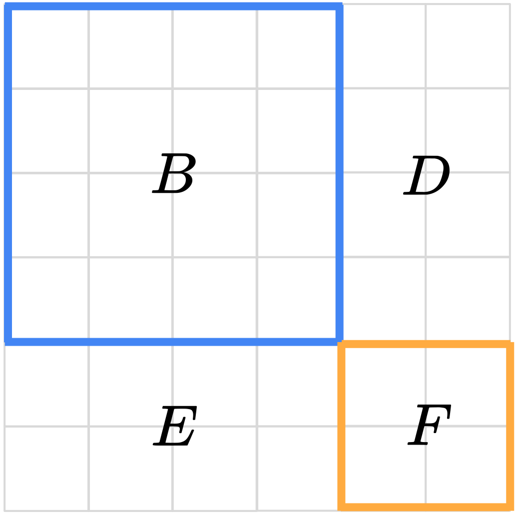
Solve:

$$\det\left(I + \frac{YX}{t^2}\right) = 0$$

$$X = DFE$$
$$Y = (B - tI)^{-1}$$

$$\det\left(I + \frac{YX}{t^2}\right) = 1 + \frac{1}{t^2}\,\frac{v_1^T X u_1}{t-\lambda_1} \iff t^2(t - \lambda_1) + v_1^T X u_1 = 0$$

Solving for eigenvalues

$B^c =$ 

Solve:

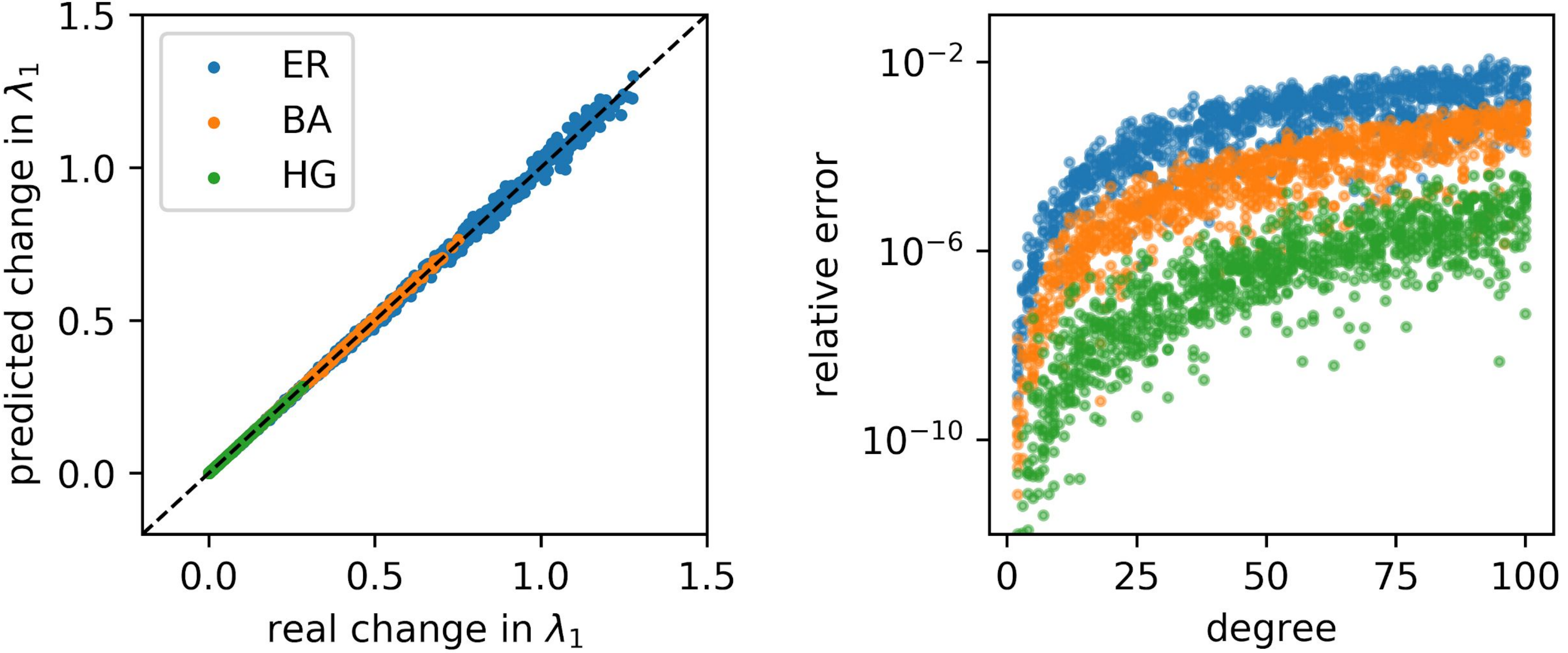
$$t^2(t - \lambda_1) + v_1^T X u_1 = 0$$

$$X = DFE$$

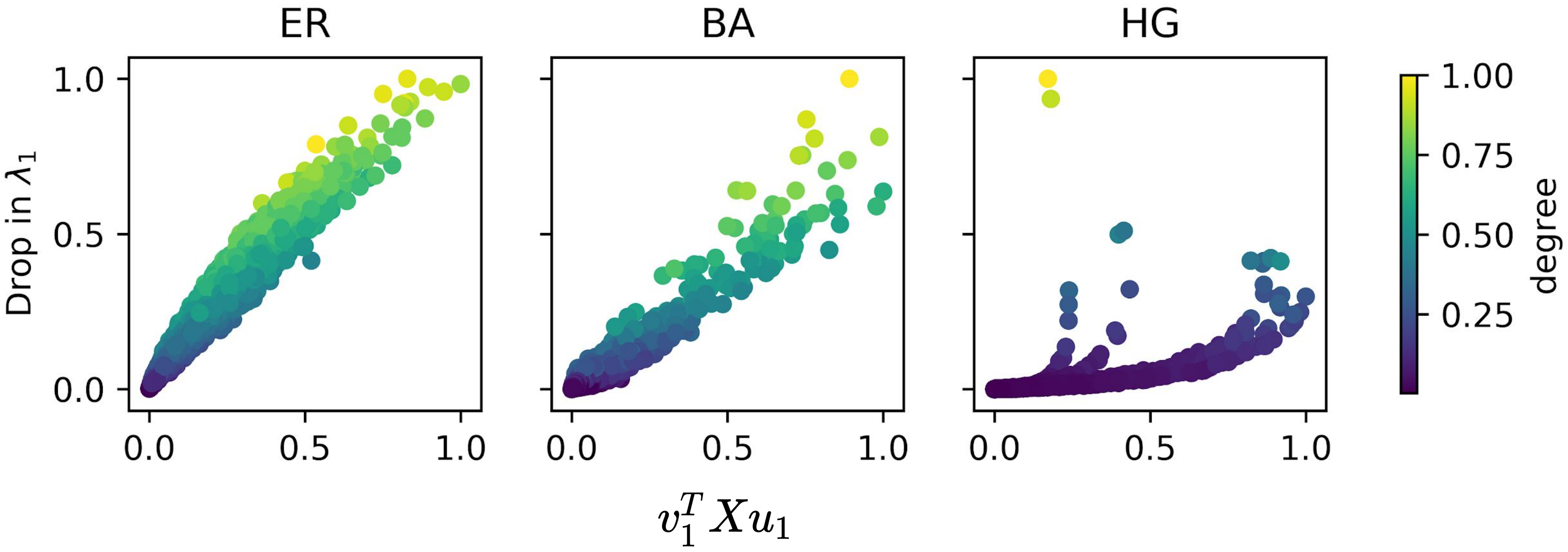
λ_1, u_1, v_1

1. Third degree polynomial: closed form solution
2. Solution increasing in $v_1^T X u_1$

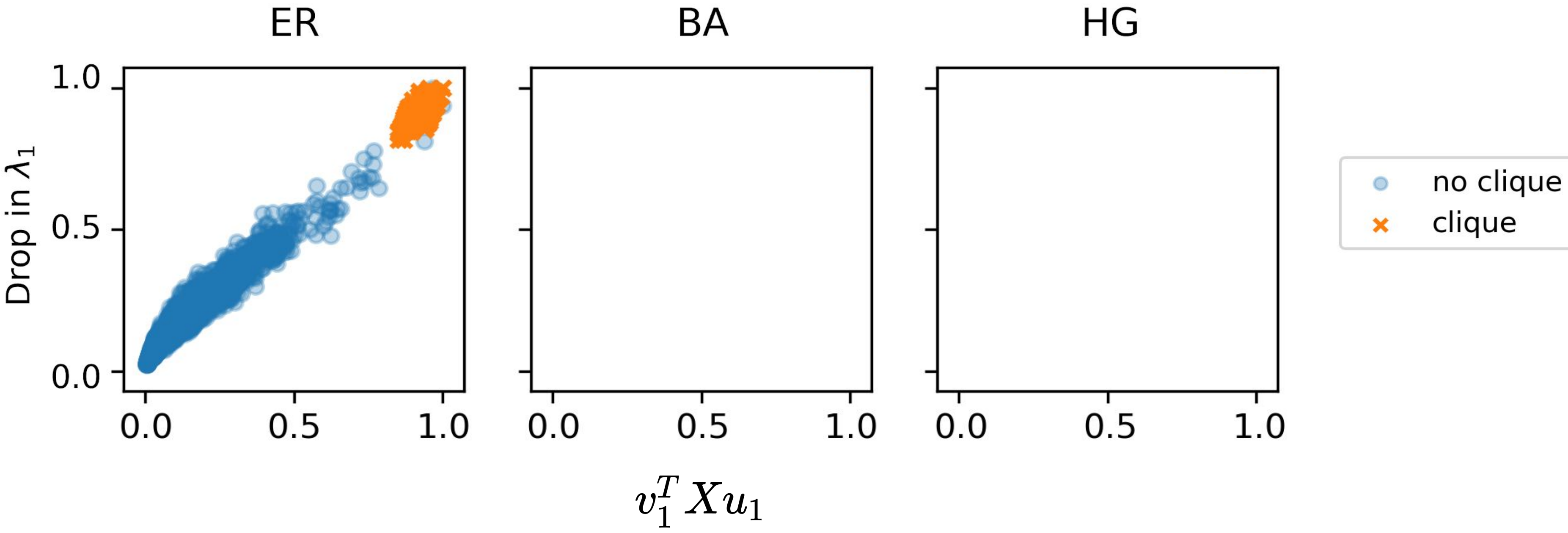
Really good approximation



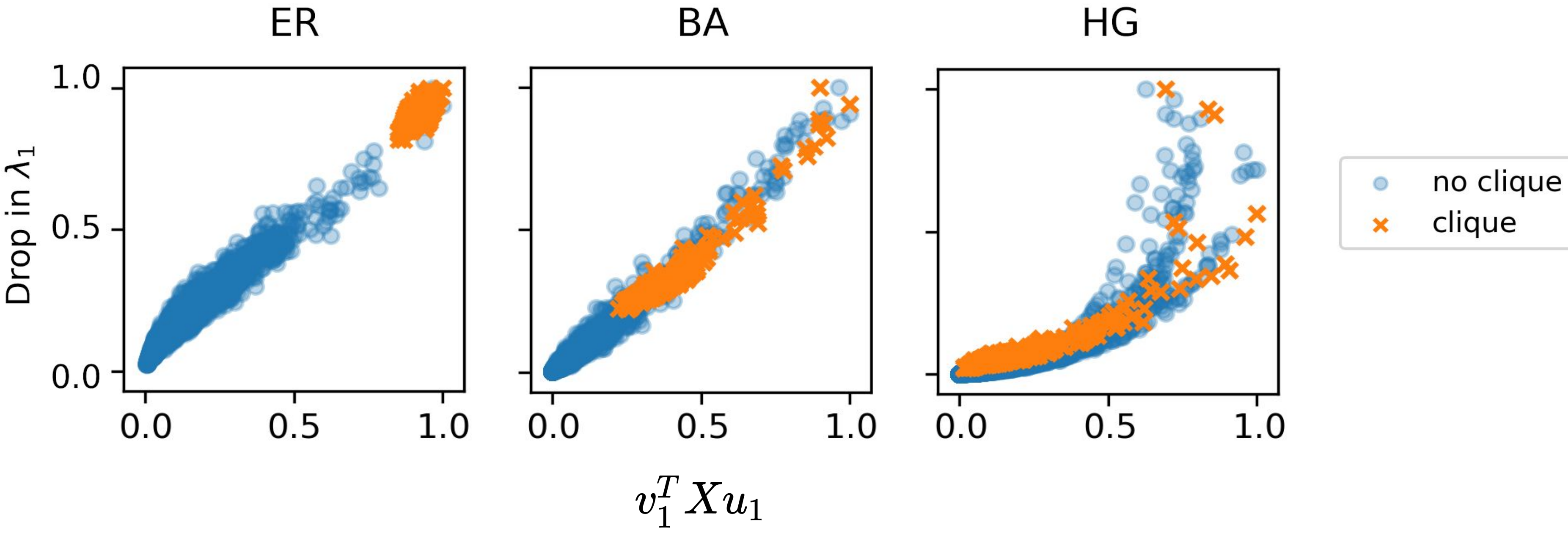
Studying the constant



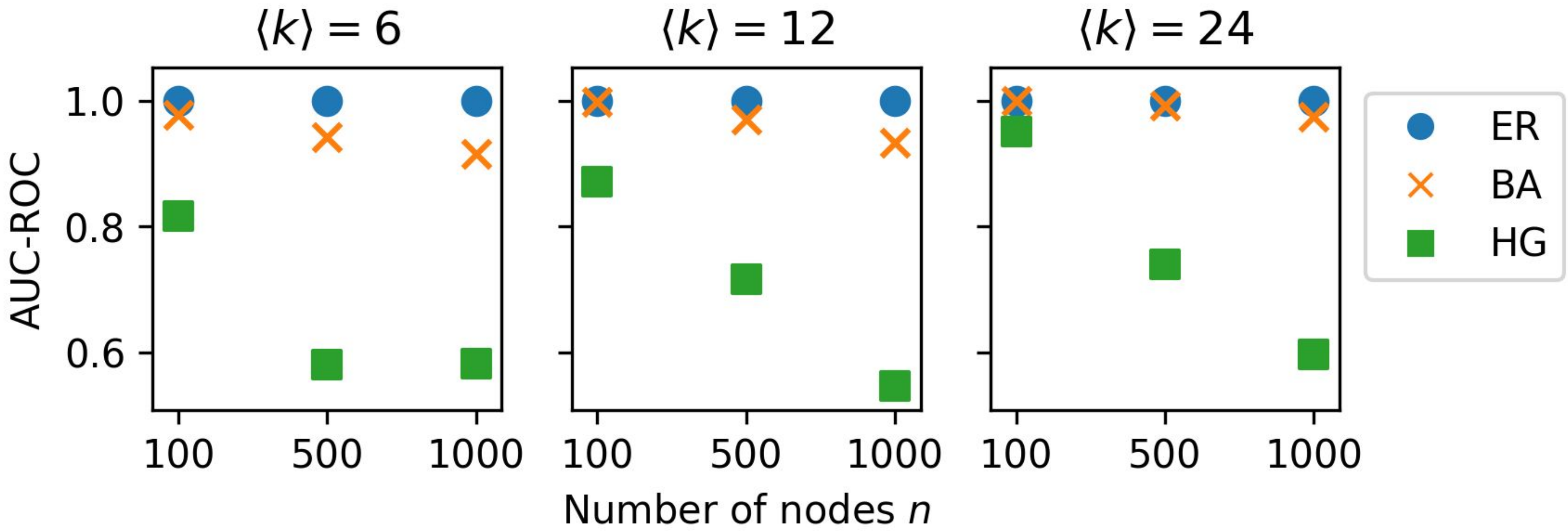
Studying the constant: planted clique



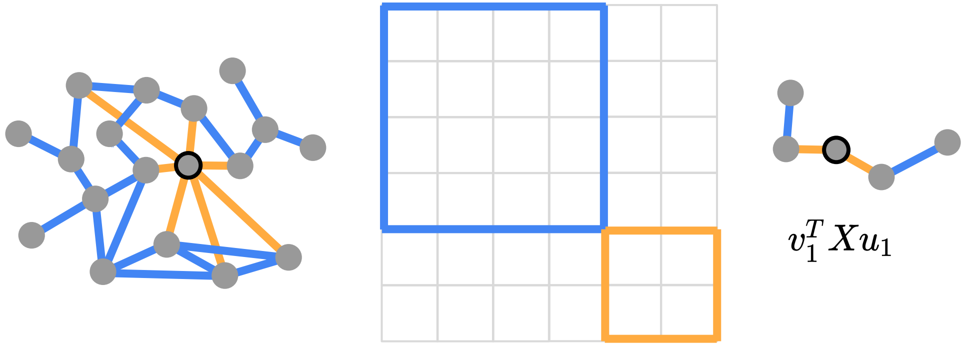
Studying the constant: planted clique



Application: clique detection



Gracias!



1. Immunization: remove hubs, break up cliques
2. Towards non-backtracking eigenvalue interlacing
3. Bounds on graph distance (NBD)