Unitary non-backtracking eigenvalues

Leo Torres

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Leo Torres
PhD in Network Science
Network Science Institute, Northeastern University
leo@leotrs.com

Non-backtracking Matrix

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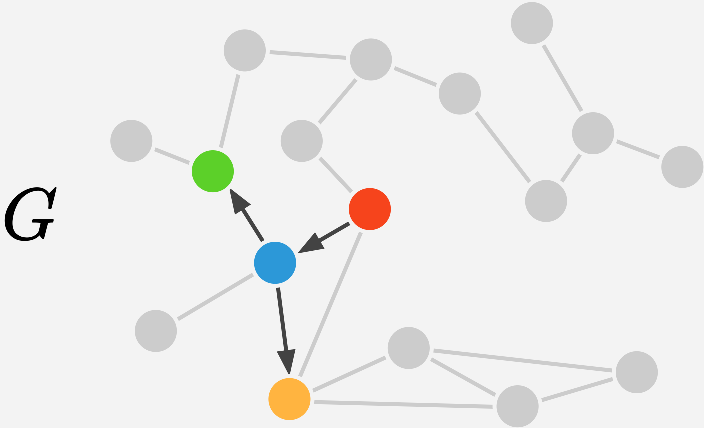
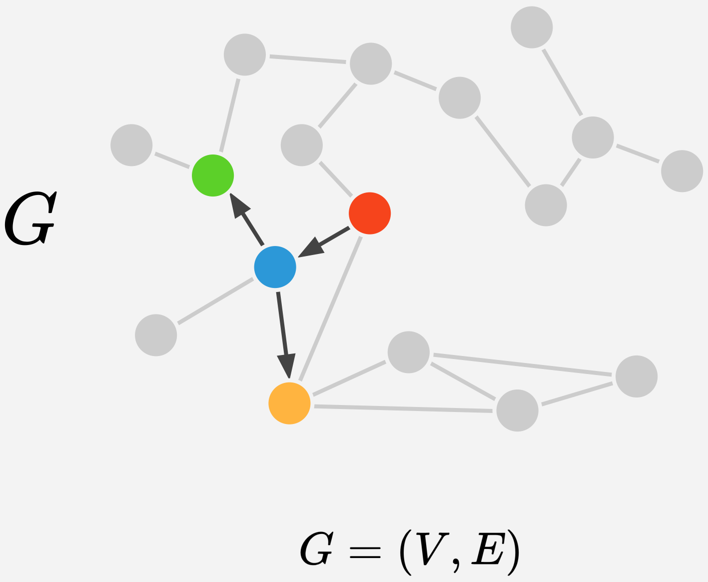
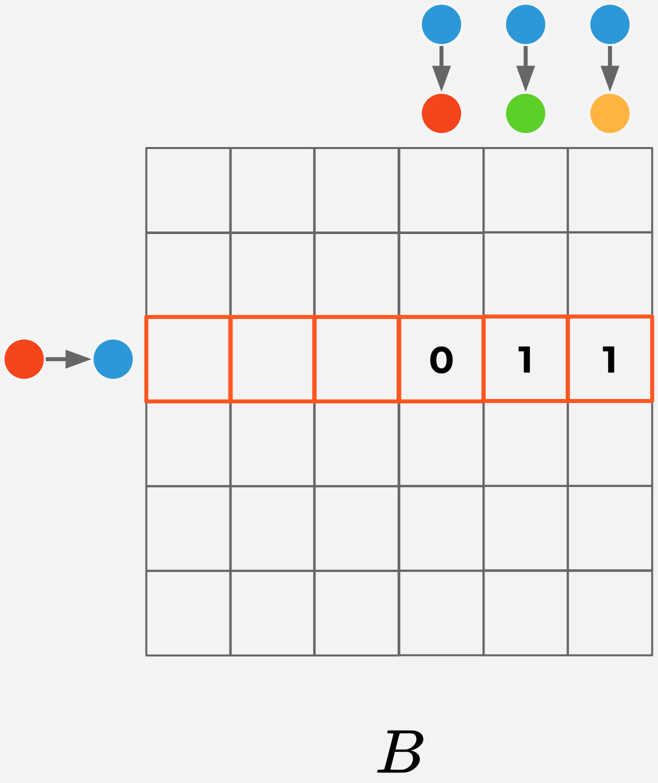


$$G = (V, E)$$

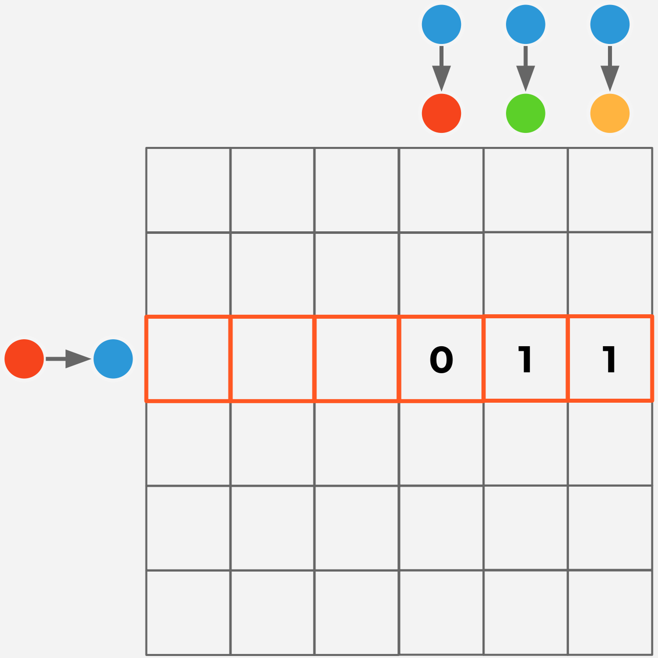
Non-backtracking Matrix





Non-backtracking Matrix

- community detection
  - Krzakala, et al. PNAS 110.52 (2013): 20935-20940.
  - Bordenave, et al. FOCS (2015).
- centrality
  - Martin, et al. Phys. Rev. E 90.5 (2014): 052808.
  - Morone & Makse. Nature 524.7563 (2015): 65-68.
  - Arrigo, et al. J. of Sci. Comp. 80.3 (2019): 1419-1437.
- dynamics (SIR, SIS)
  - Karrer, et al. Phys. Rev. Lett. 113.20 (2014): 208702.
  - Hamilton, & Pryadko. Phys. Rev. Lett. 113.20 (2014): 208701.
  - Shrestha, et al. Phys. Rev. E 92.2 (2015): 022821.
  - Castellano, & Pastor-Satorras. Phys. Rev. E 98.5 (2018): 052313.
  - Masuda, et al. J. of App. Math. 85.2 (2020): 214-230.



B

Anecdotal observation:

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WHY?

Questions

- What's different about the unitary eigenvalues?
- Can we compute their multiplicities?
- Is the geometric mult. equal to the algebraic mult.?
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- How does this affect the diagonalizability of the matrix?

related to certain subgraphs

yes!

maybe…

it (maybe) doesn't!

Setting

Let G be a graph with non-backtracking matrix B. Let $B\mathbf{v} = \lambda\mathbf{v}$ and $|\lambda| = 1$.

Tasks:

- study $\mathbf{v}$, λ
- compute the number of such $\mathbf{v}$ that are L.I.

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Recall:

- B has one row/column for each oriented edge (size $2m \times 2m$)
- $\mathbf{v}$ is a function of the oriented edges

Some initial facts

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- As a matrix:

$$B_{k\to l,i\to j} = \delta_{jk}\left(1 - \delta_{il}\right)$$

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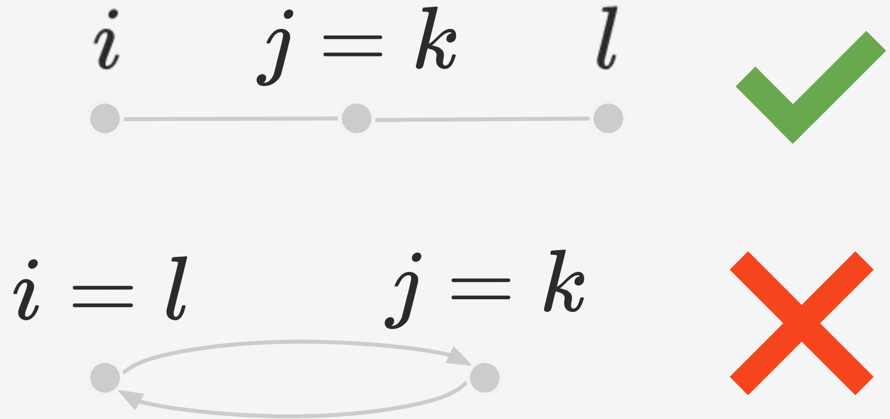

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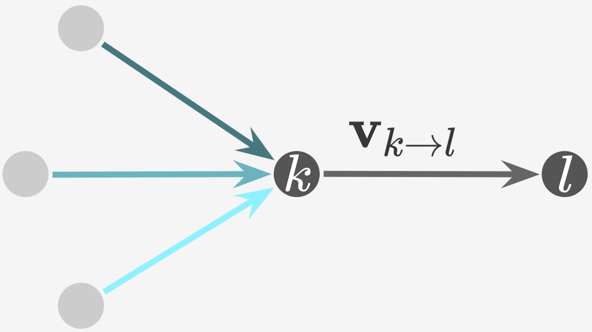

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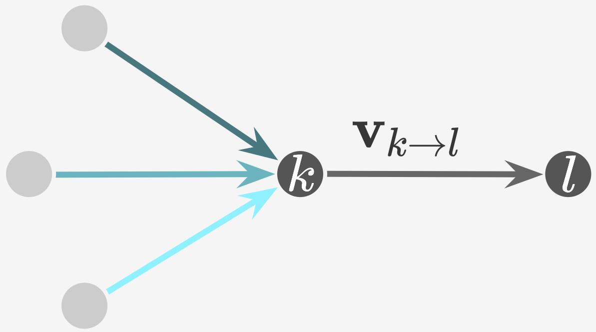
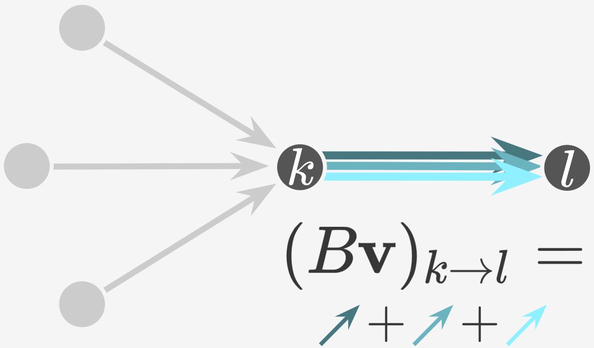


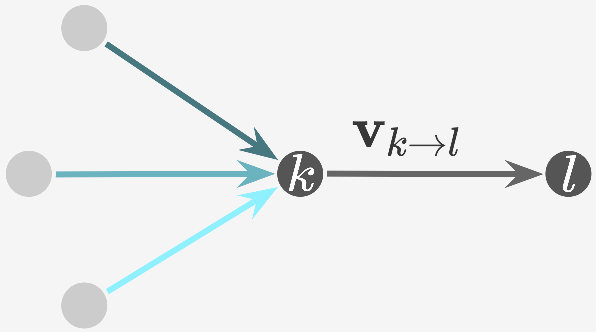


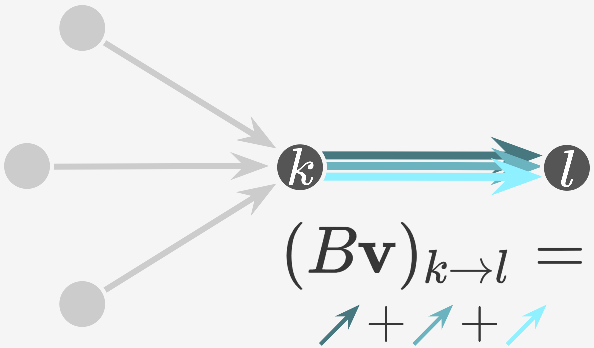
Some initial facts

- As an operator:

"v into k"

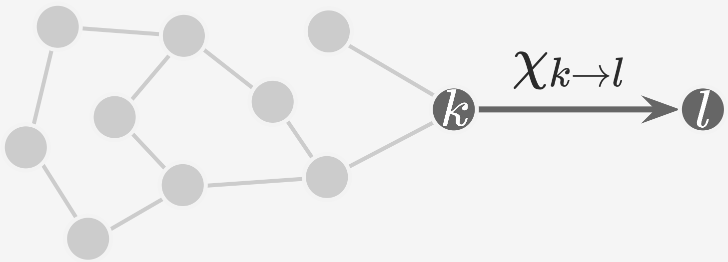
$$(B\mathbf{v})_{k\to l} = \vec{\mathbf{v}}^{k} - \mathbf{v}_{l\to k}$$





Some initial facts

- Take the characteristic function of a directed edge pointing to a node of degree one:



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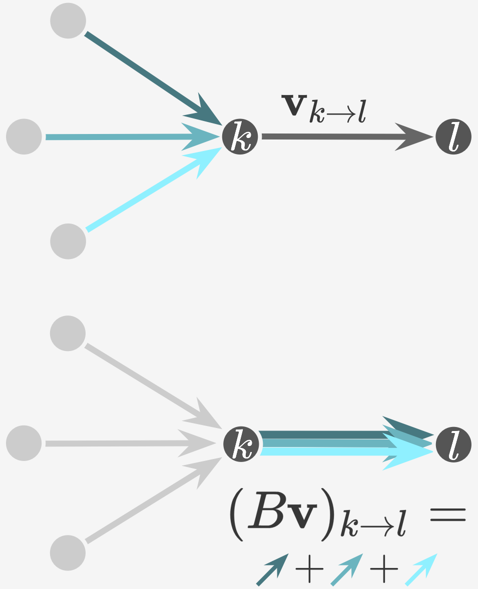

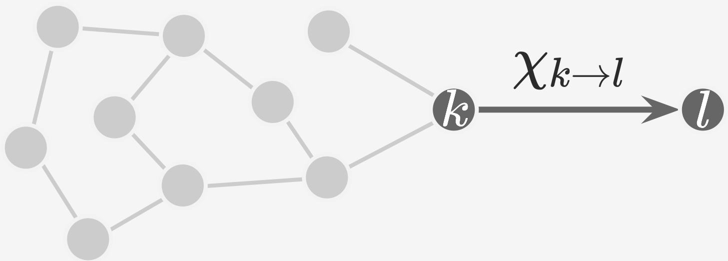

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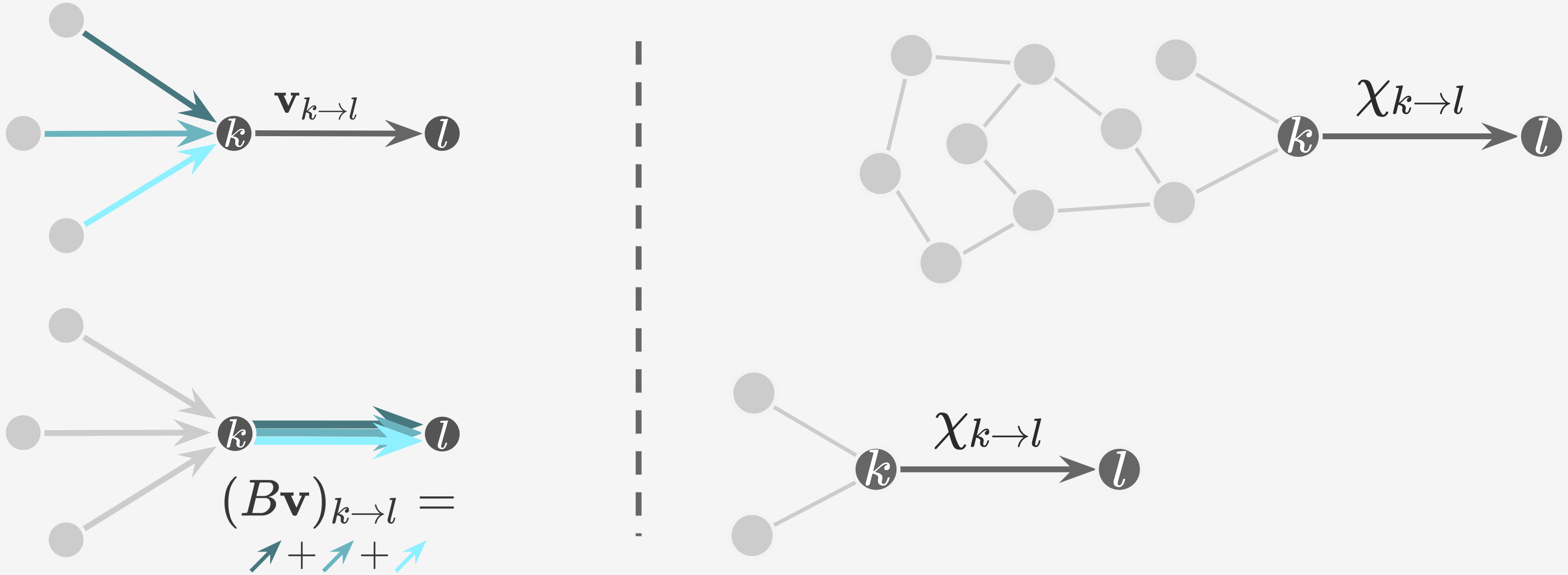

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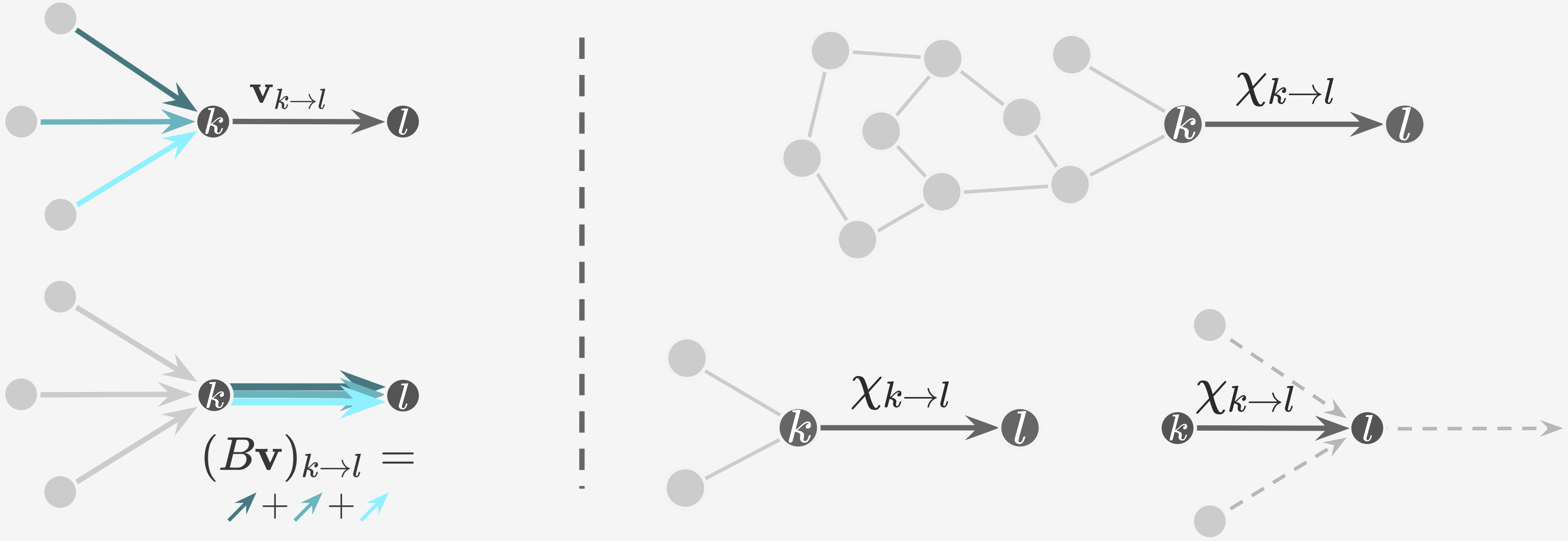

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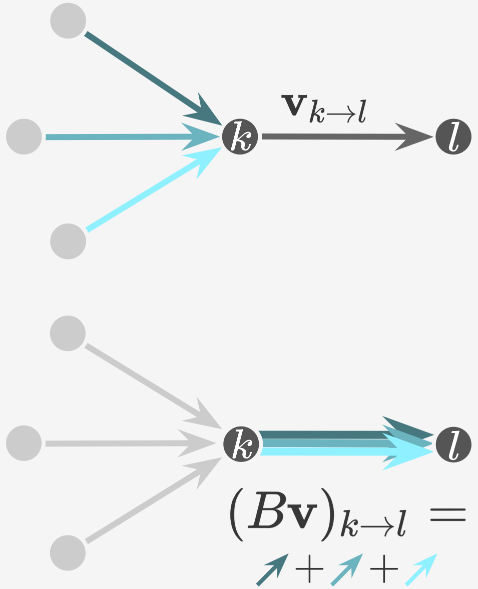

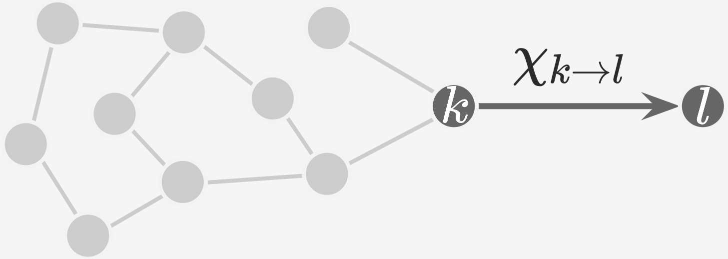

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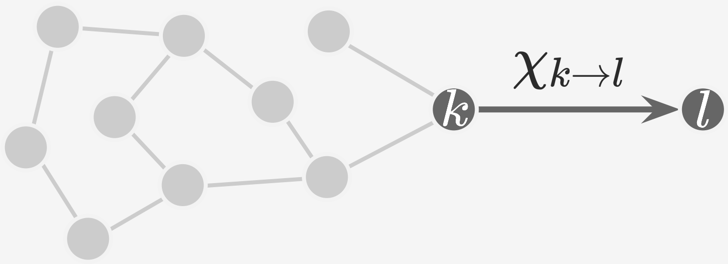
What's more: it can be shown that nodes of degree one do not influence the non-zero eigenvalues.



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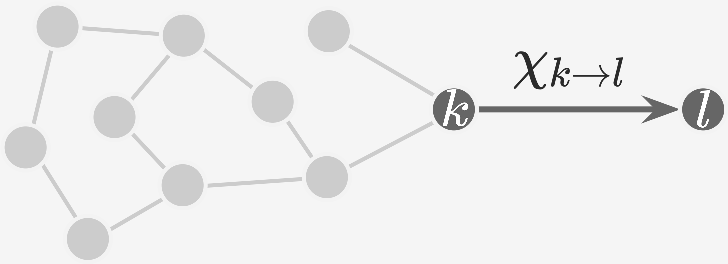
From now on, assume the graph has minimum degree 2.



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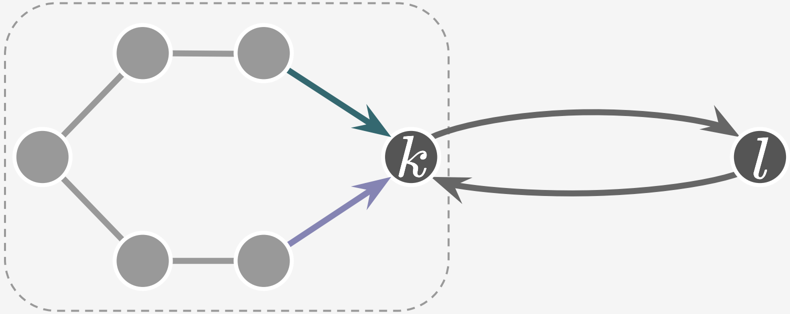

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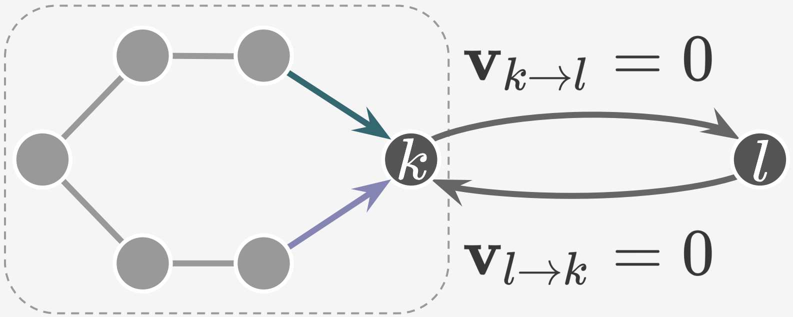

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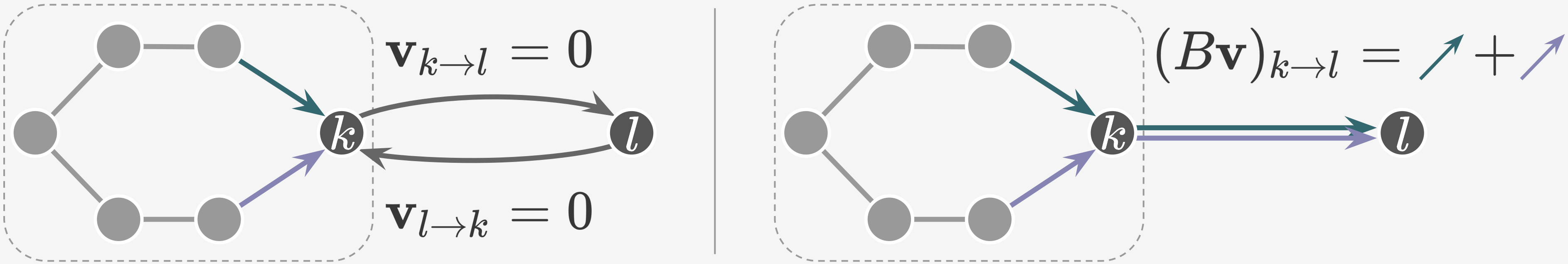

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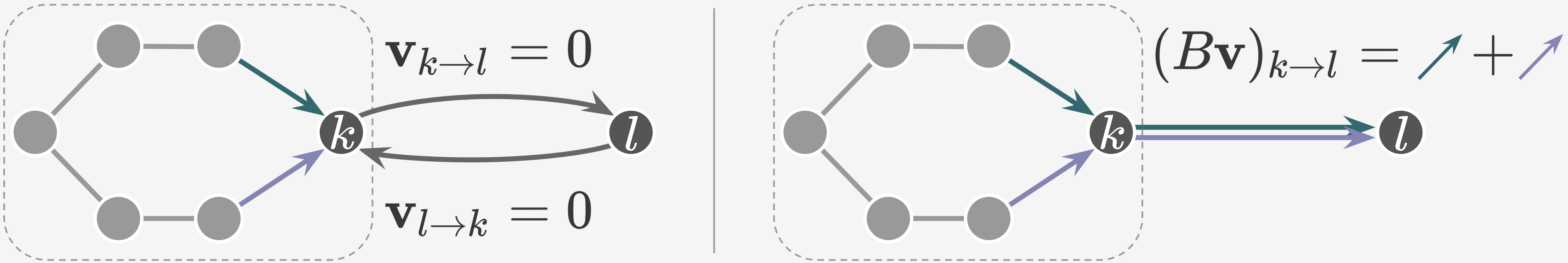

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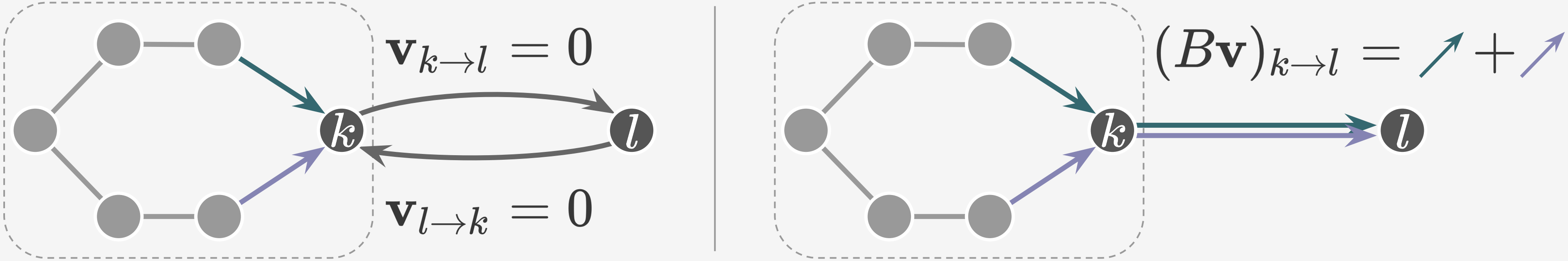

$$\vec{\mathbf{v}}^k = 0 \quad \text{or} \quad d_k = 2$$

Some initial facts: recap

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- For any e'vector of unitary e'value:

$$(d_l - 2)\,\vec{\mathbf{v}}^{l} = 0,\ \forall l$$

Unitary e'values are roots of unity

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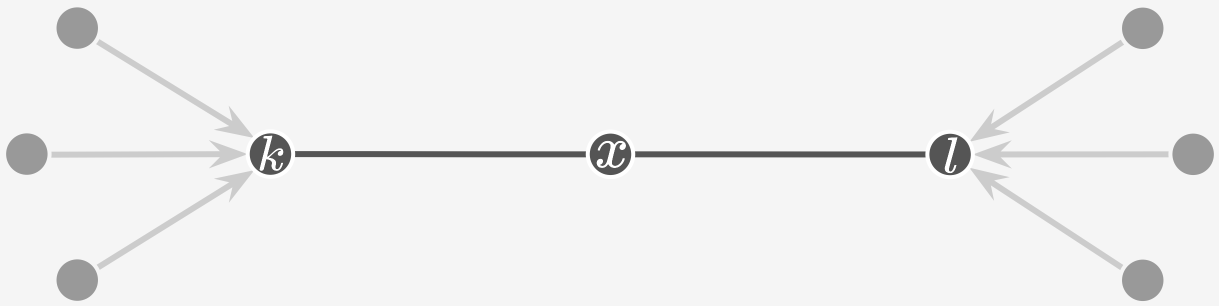

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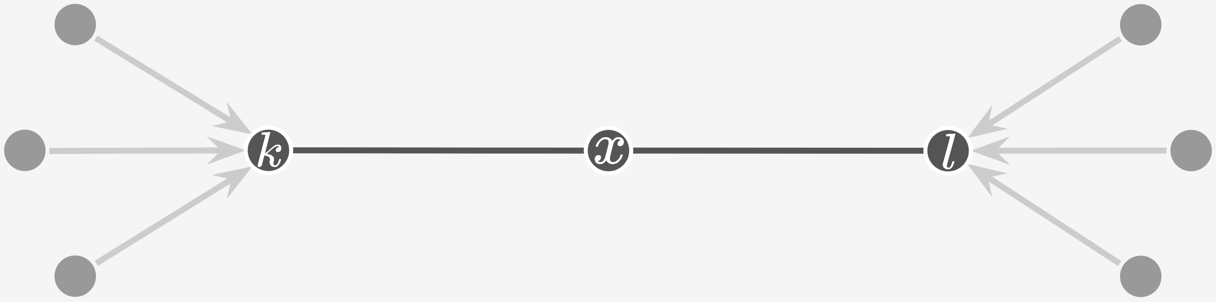

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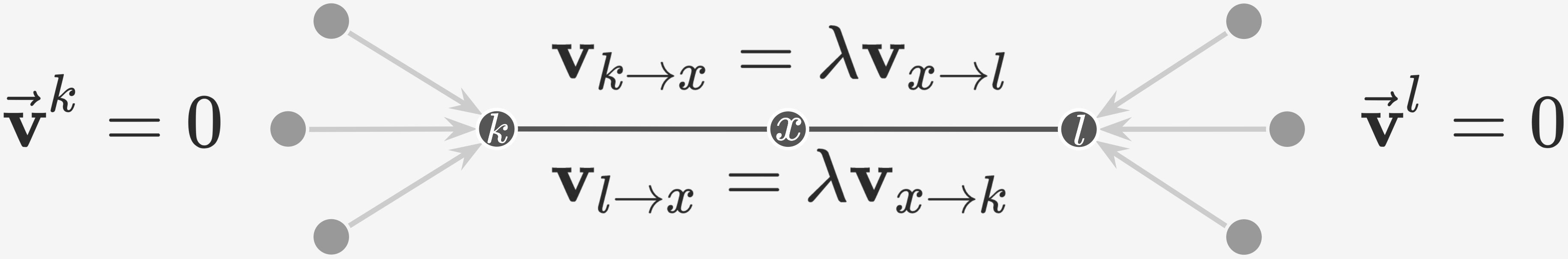

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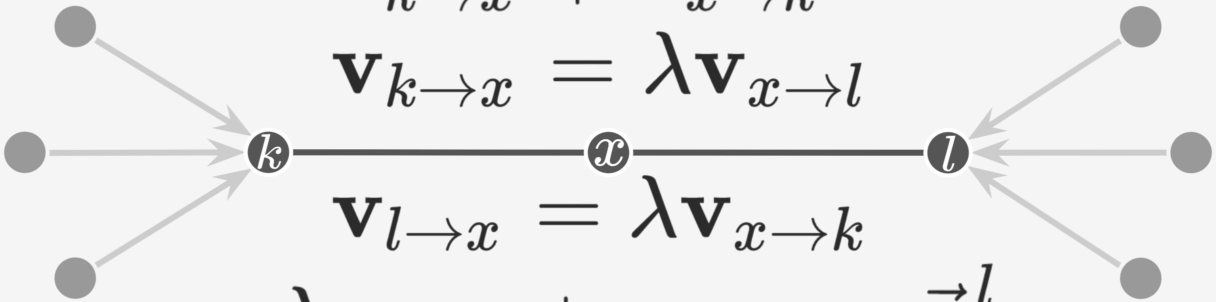

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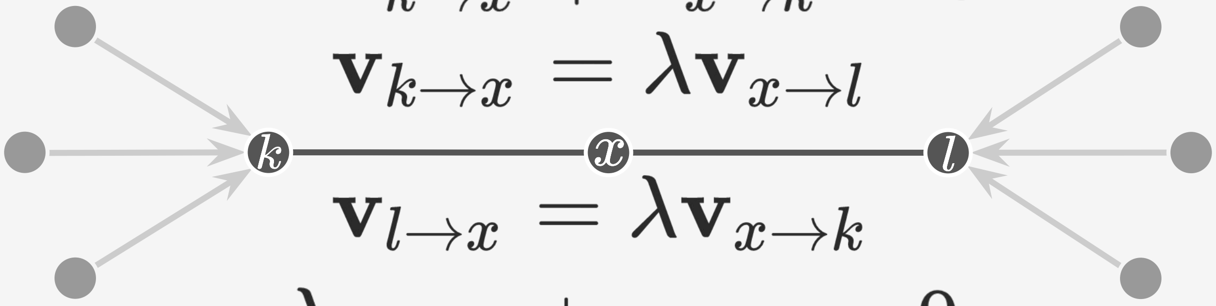

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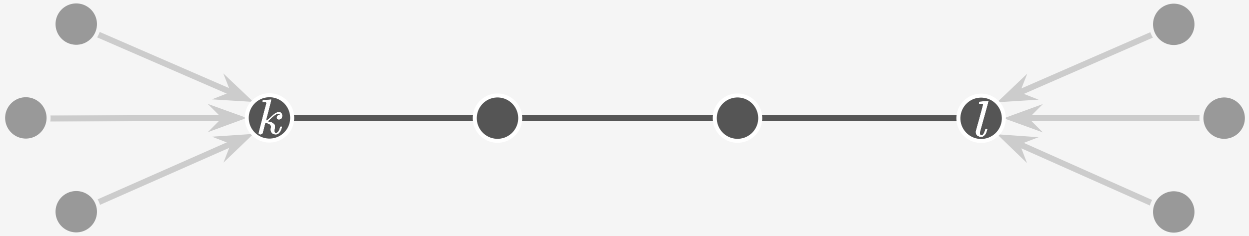

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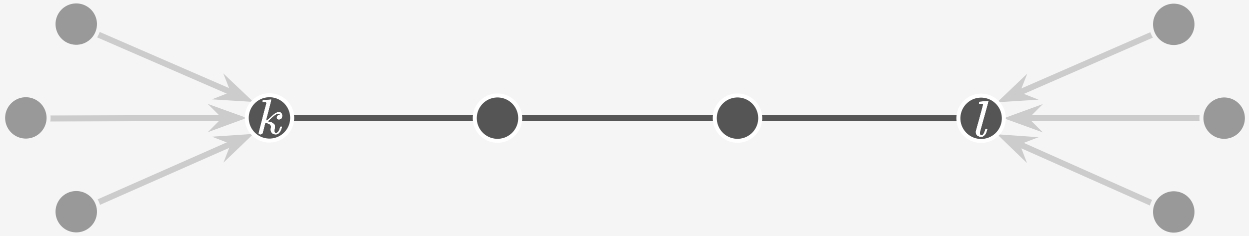

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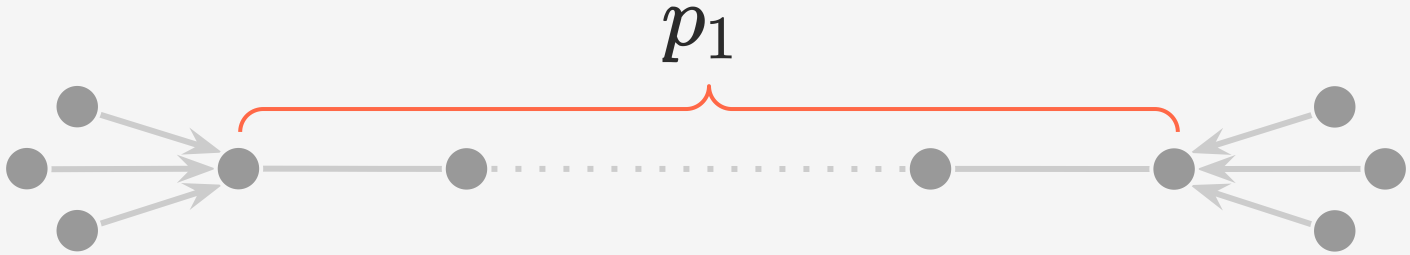
More assumptions...

From now on:

1. finite
2. simple
3. undirected
4. unweighted
5. connected
6. minimum degree 2
7. not a circle graph (i.e. at least one node w degree 3+)

Support of a unitary e'vector



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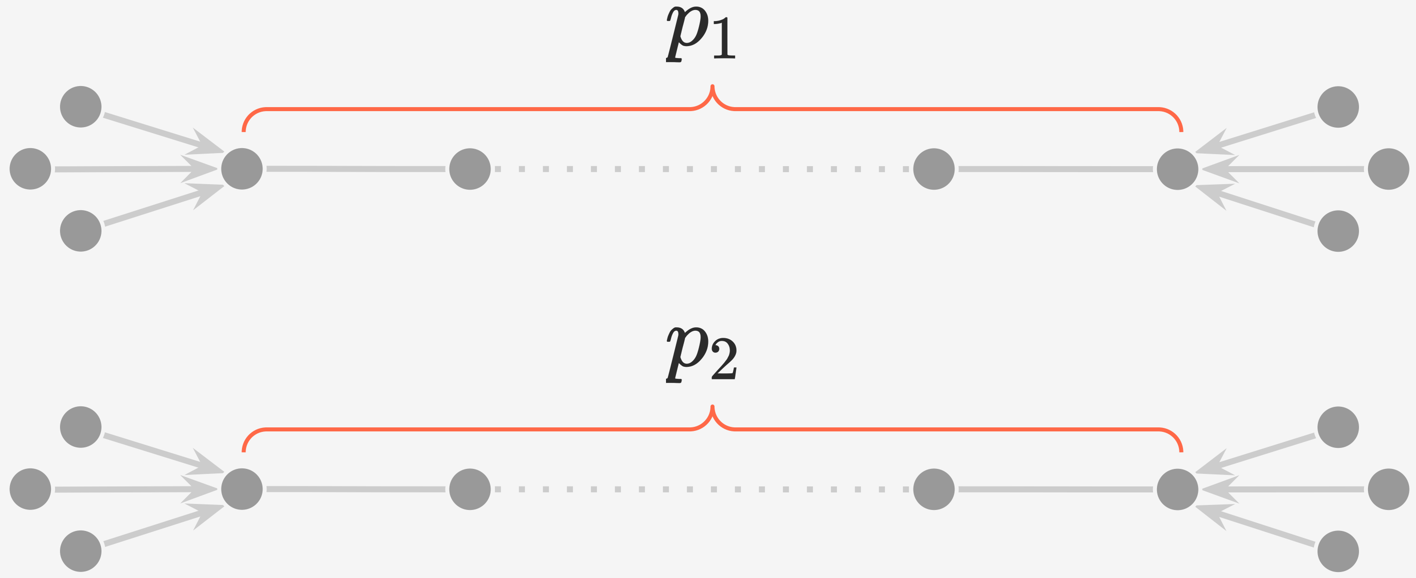

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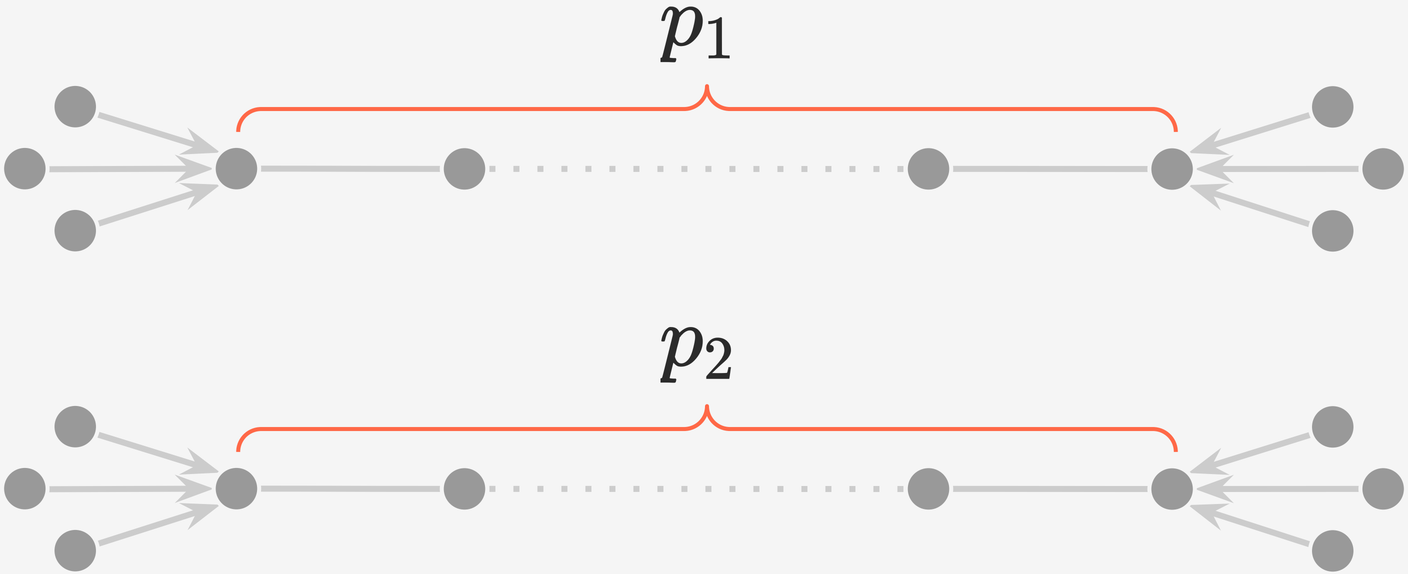

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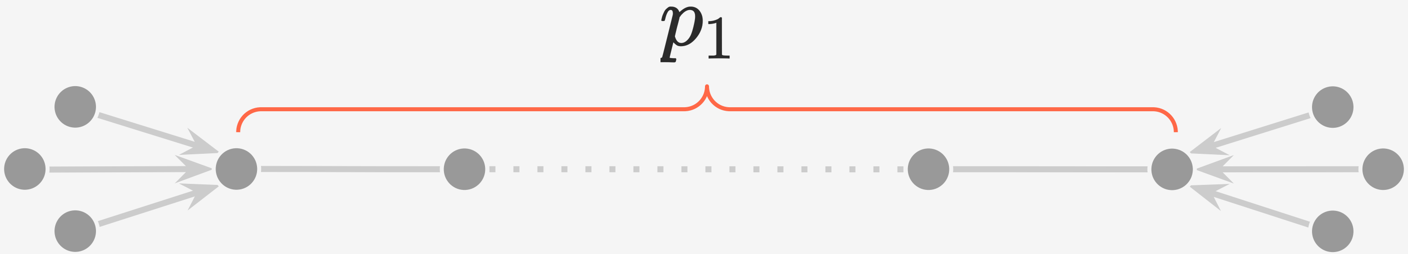

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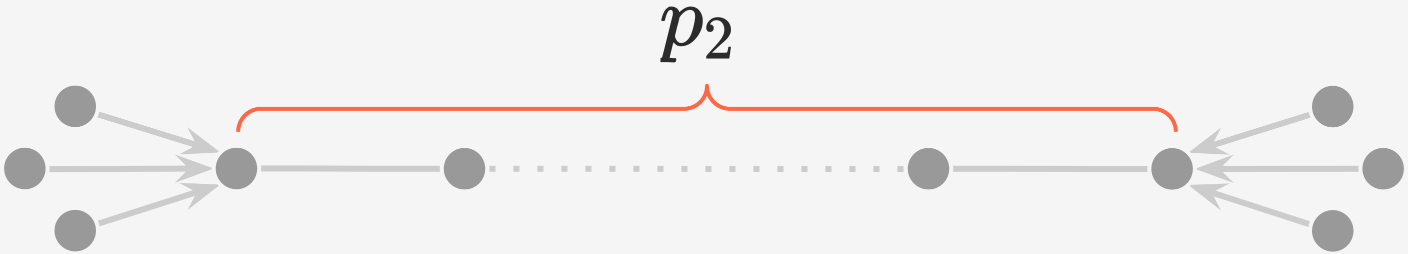

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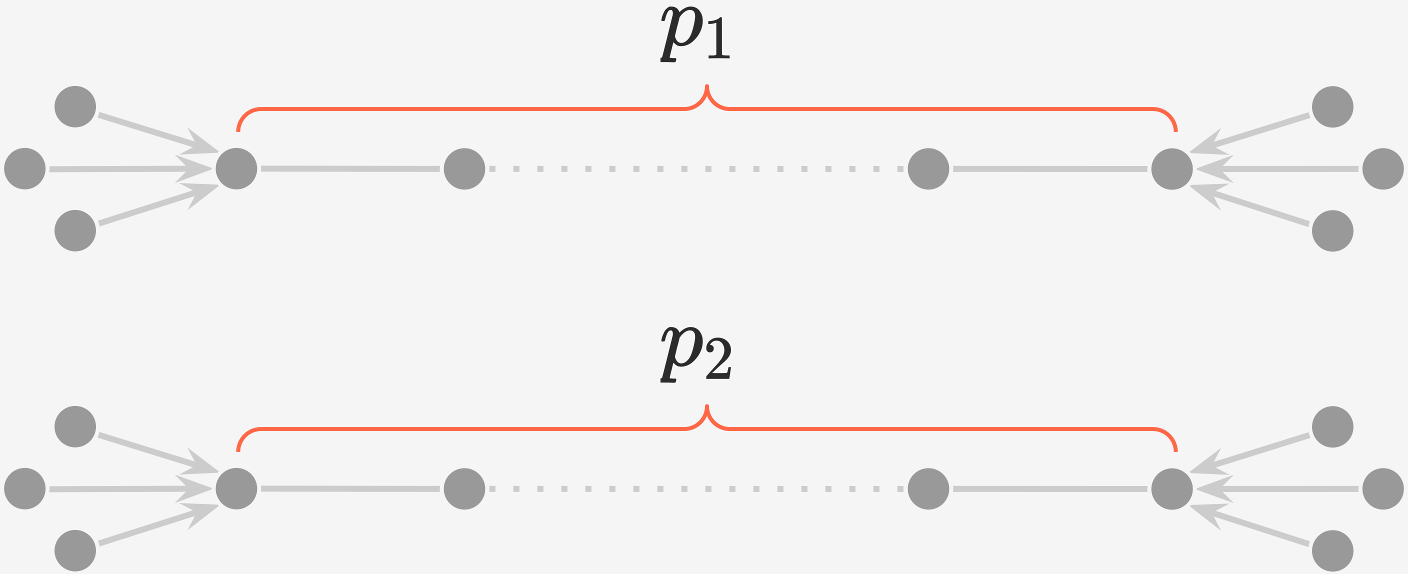

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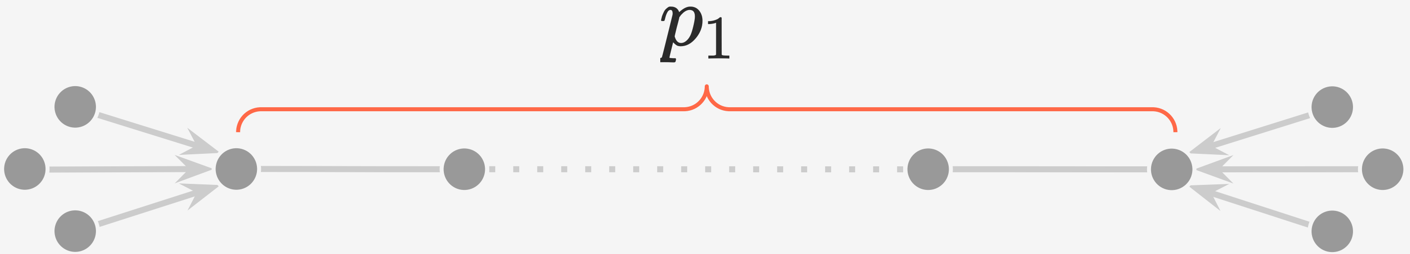

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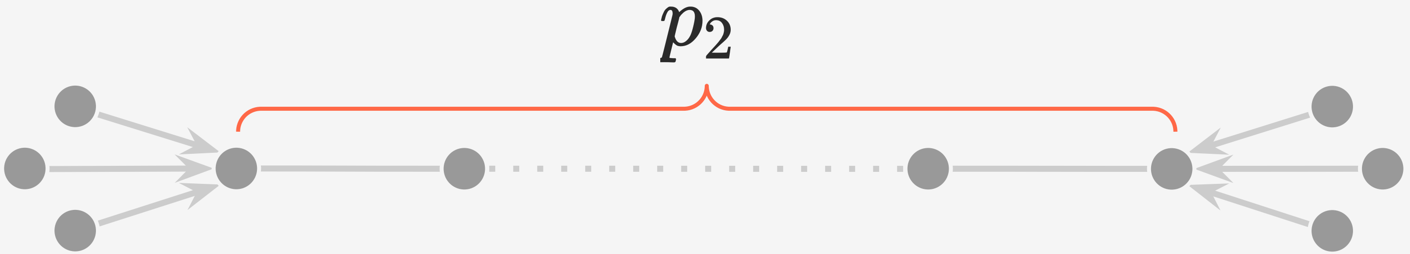

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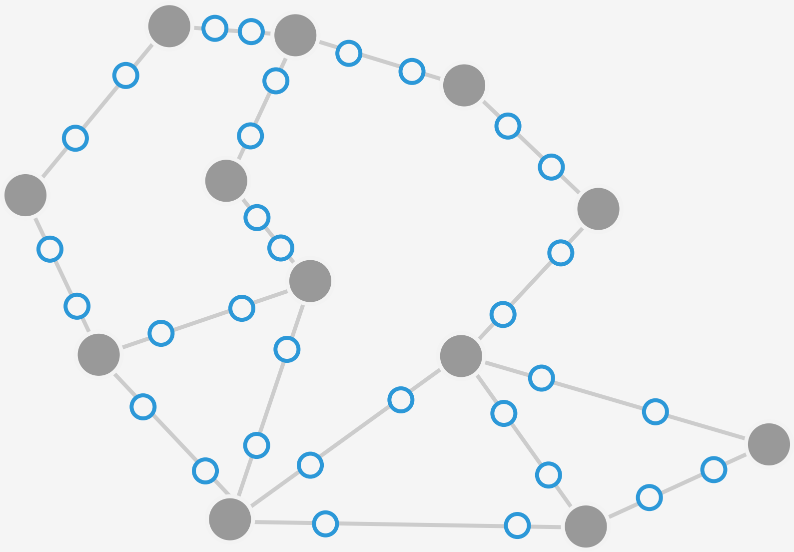
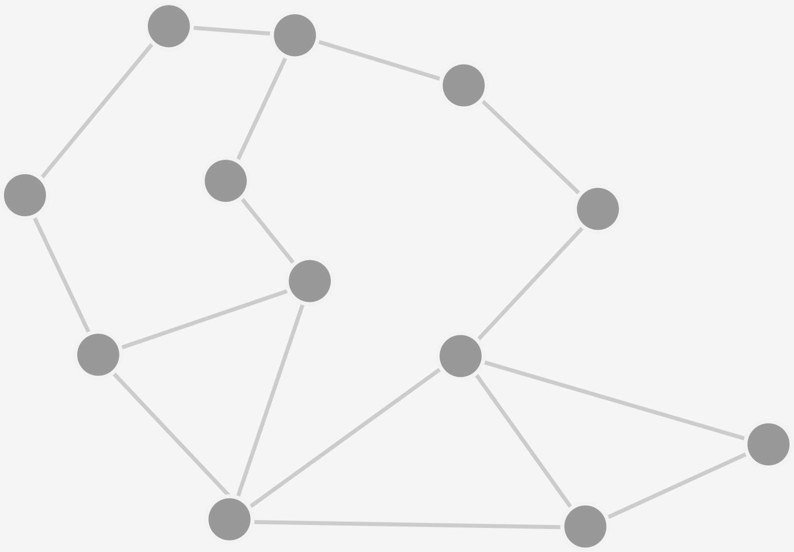
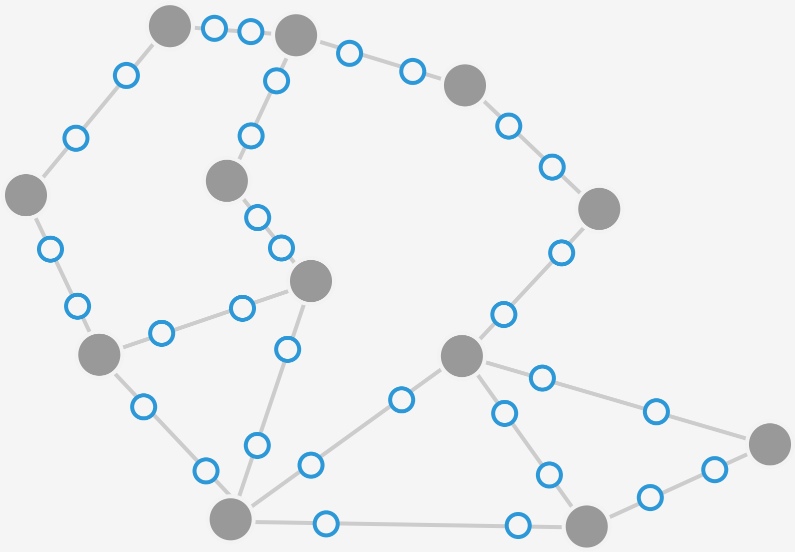


Graph subdivisions

Definition: For a graph G, its p-th subdivision is the graph formed by replacing each edge by a chain of length p.





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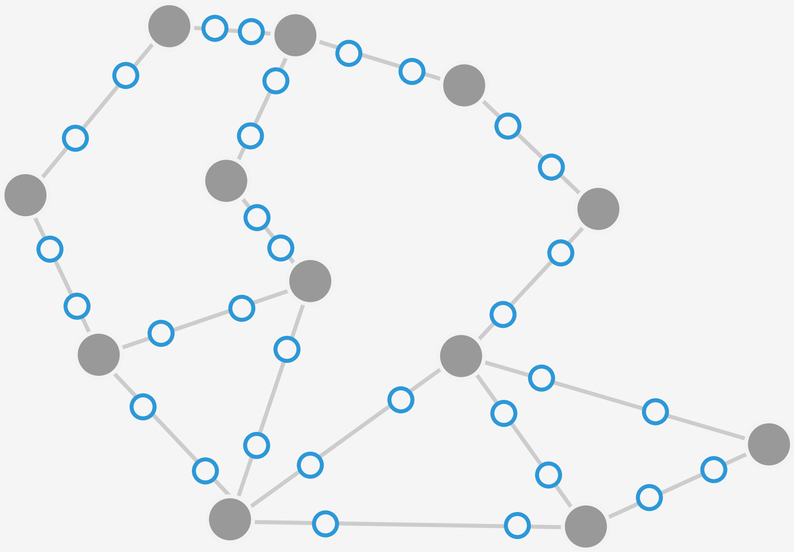
Theorem: Suppose $\lambda^{2p} = 1, p \neq 1$, then its support must be the p-th subdivision of some graph.



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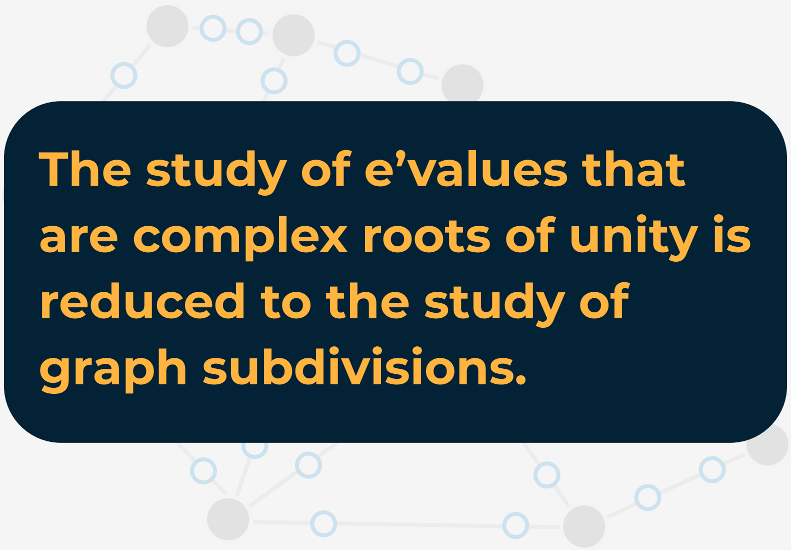

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This is what's special about unitary e'values.

E'vectors of subdivisions

Theorem: Let H be the p-th subdivision of some graph G s.t. G is not the subdivision of any other graph. Suppose λ is a p-th root of unity. Then $GM(\lambda) = |E| - |V| + 1$.

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$$Z_G(t) = \prod_{c \in G} \left(1 - t^{|c|}\right)^{-1}$$

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$$B_G\mathbf{v} = \lambda^p\mathbf{v} \implies \begin{aligned} B_H\mathbf{v}_1 &= \lambda_1\mathbf{v}_1 \\ B_H\mathbf{v}_2 &= \lambda_2\mathbf{v}_2 \\ &\cdots \\ B_H\mathbf{v}_p &= \lambda_p\mathbf{v}_p \end{aligned}$$

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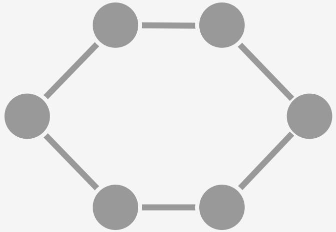

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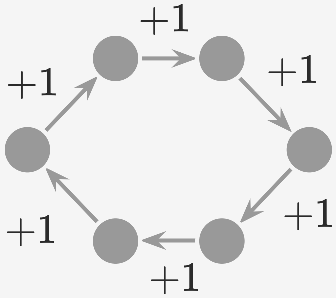

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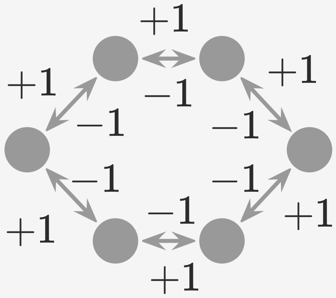

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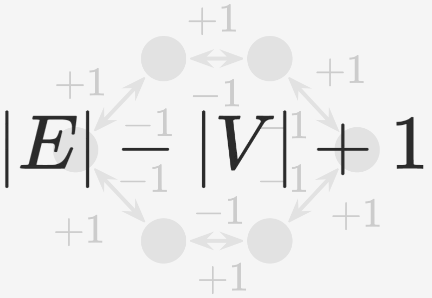

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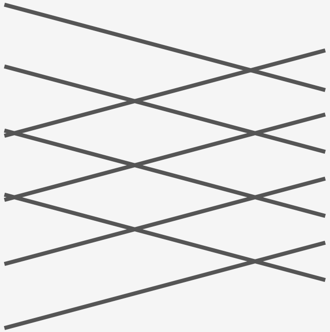
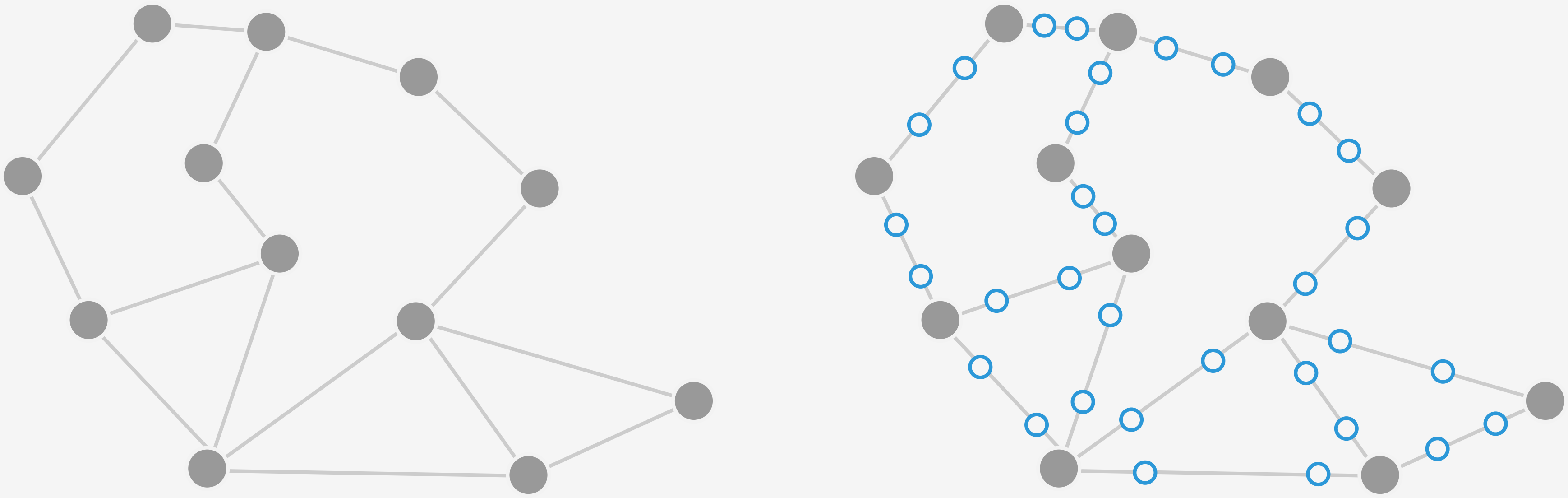
$$H = G$$



$$S_p$$

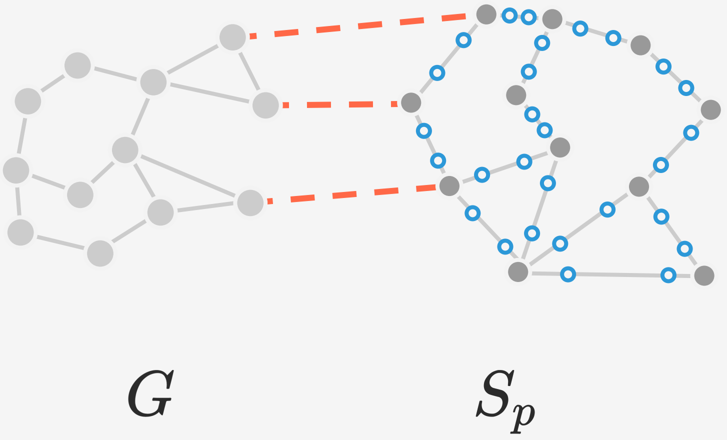
Some notation

Definition: True nodes and subdivision nodes.



Gluing a subdivision

Identify some of the true nodes of S_p with a node in G.



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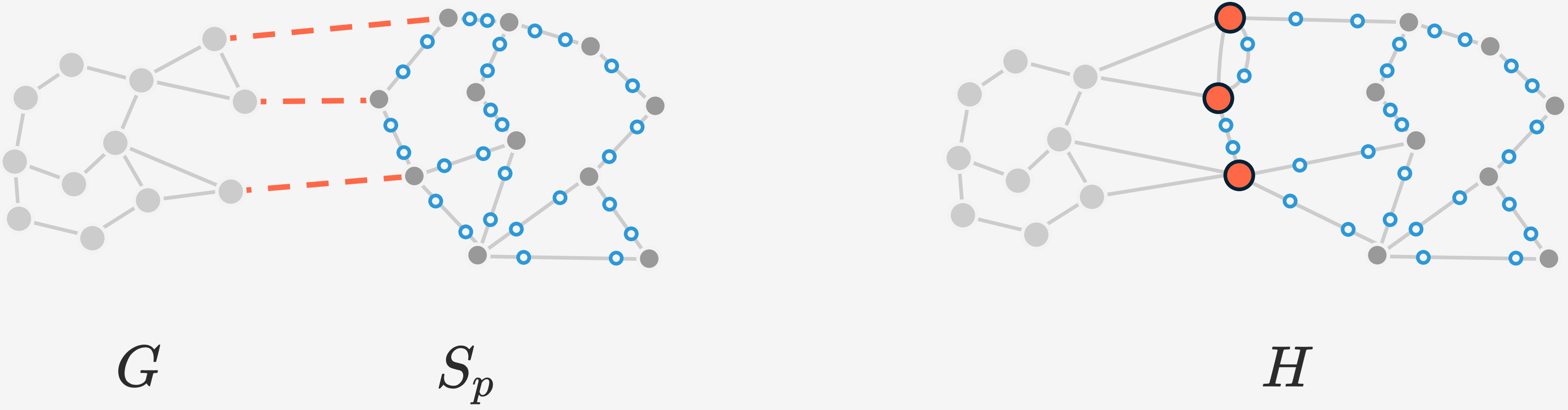

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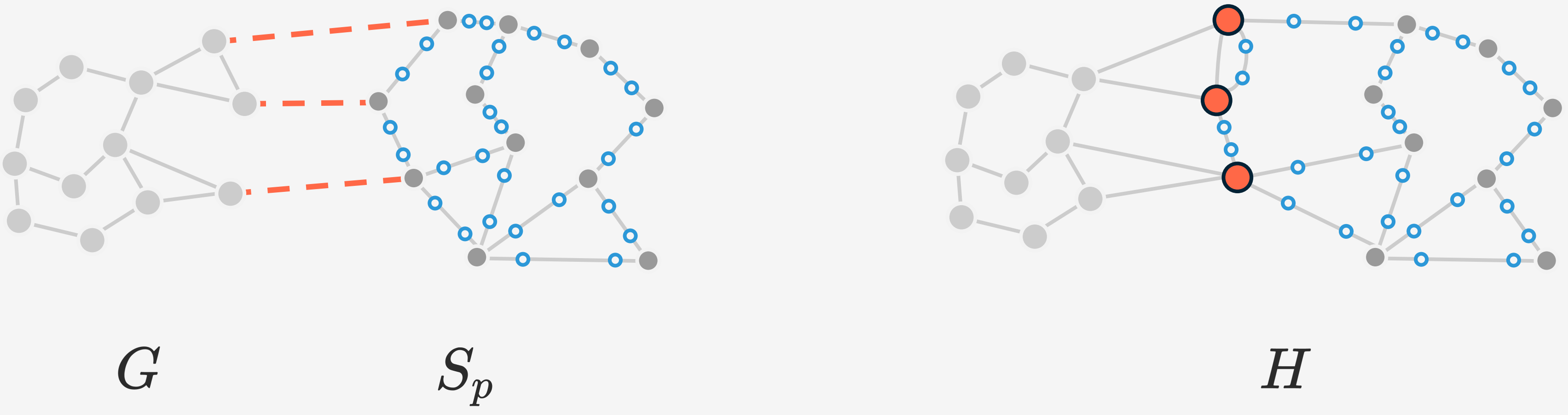
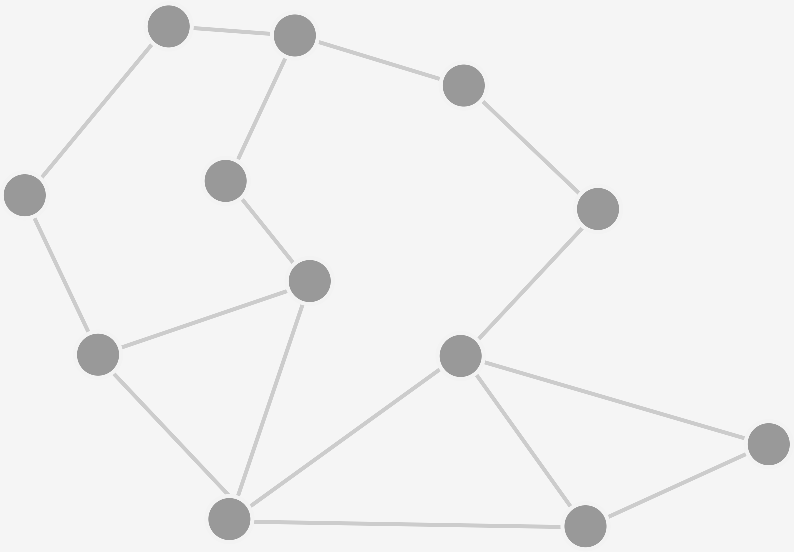
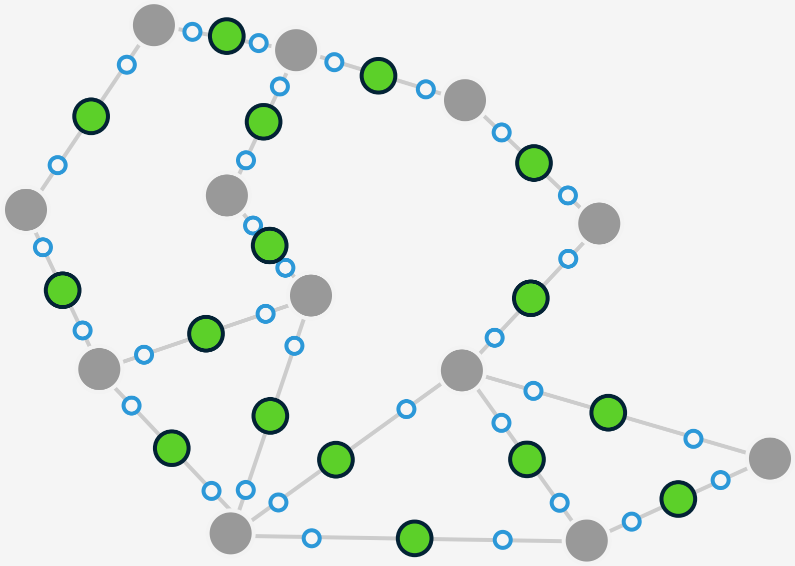
Lemma: Take an eigenvector of S_p. Pad it with zeros. Then that vector is “non-leaky” in H, i.e. $(d_l - 2)\,\vec{\mathbf{v}}^l = 0,\ \forall l$.



Some notation

Definition: if the subdivision is even, we also have middle nodes.

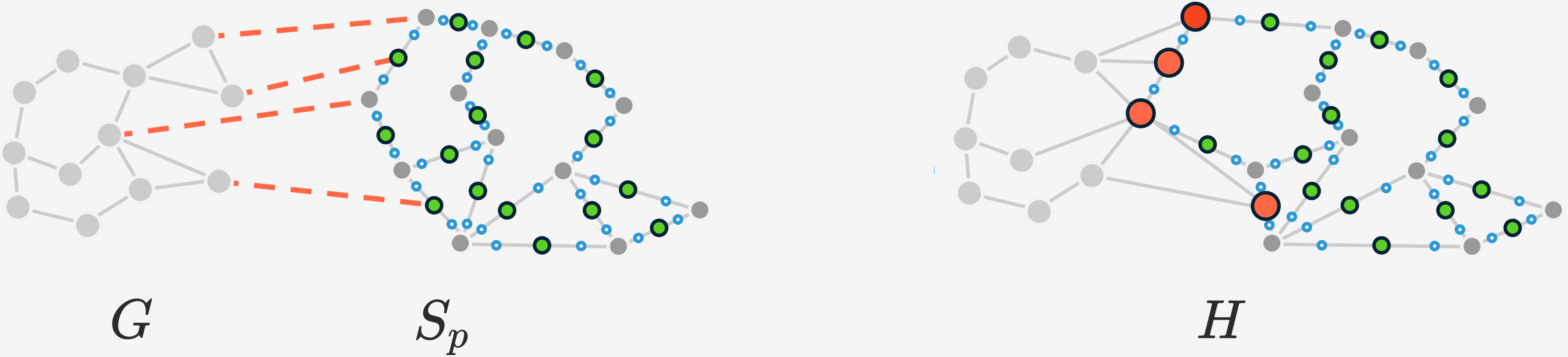




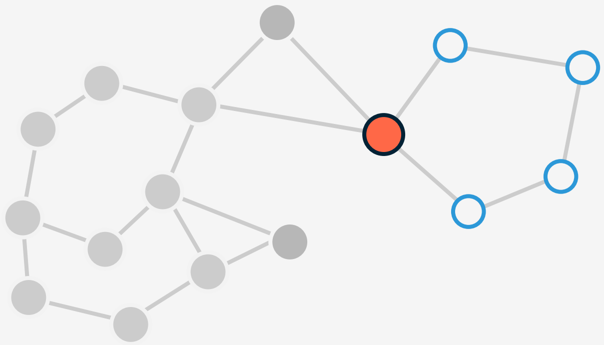
Gluing a subdivision

Identify some of the true or middle nodes nodes of S_p with a node in G.

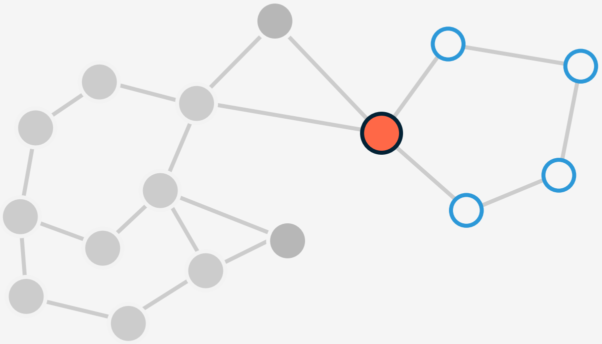
Lemma: Take an eigenvector of S_p. Pad it with zeros. Then that vector is "non-leaky" in H, i.e. $(d_l - 2)\,\vec{\mathbf{v}}^l = 0,\ \forall l.$

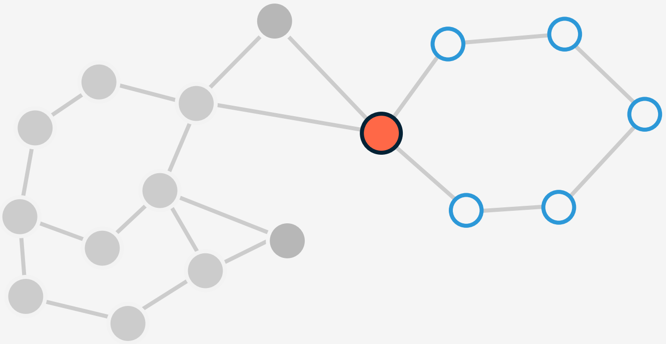


Examples

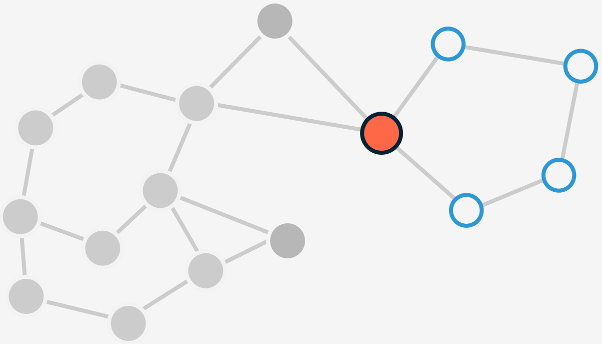


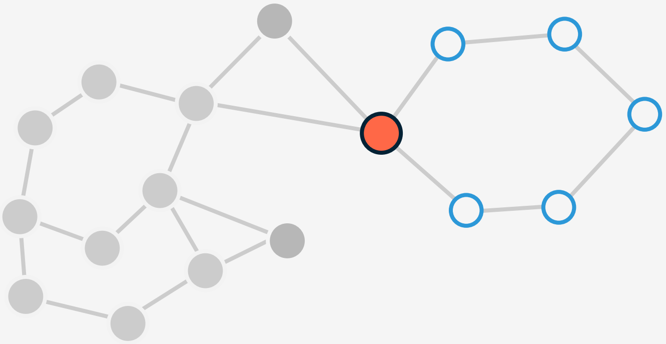
Examples

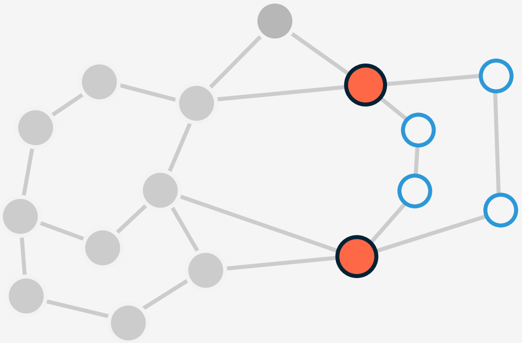




Examples







Algebraic multiplicity

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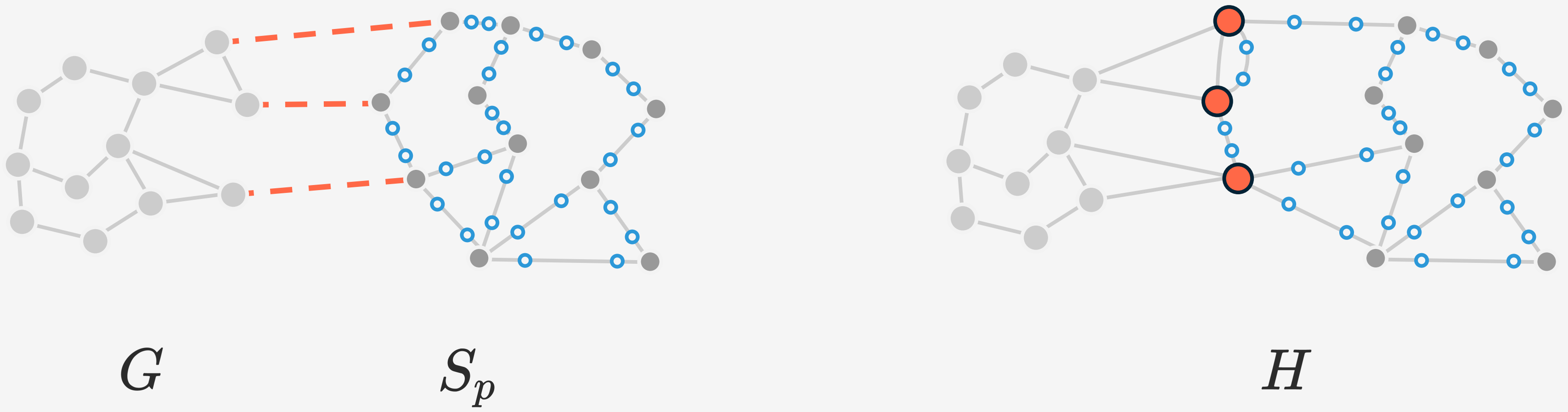
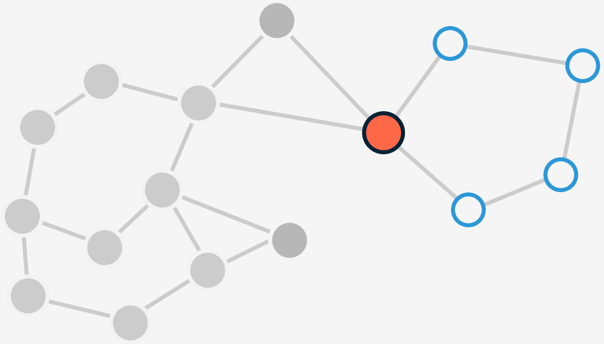


Algebraic multiplicity

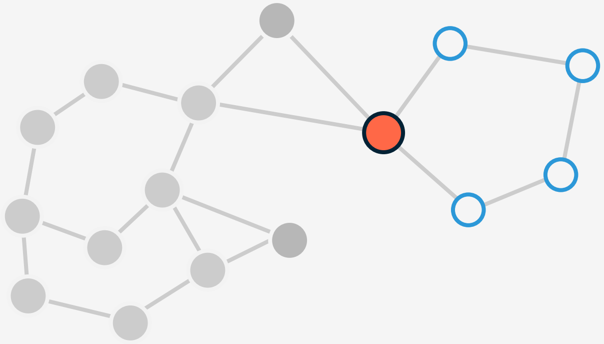
Theorem: Form H by gluing a circle graph of length k to some graph G. Then the characteristic polynomial of H satisfies:



Algebraic multiplicity

Theorem: Form H by gluing a circle graph of length k to some graph G. Then the characteristic polynomial of H satisfies:

$$\det\left(B_H - tI\right) = \left(t^{2k} - 1\right)\det\left(B_G - tI\right) - 2\left(t^k - 1\right)\det\left(B_G - C_h - tI\right)$$



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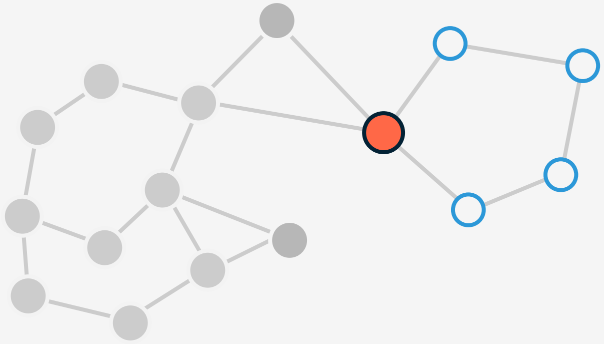
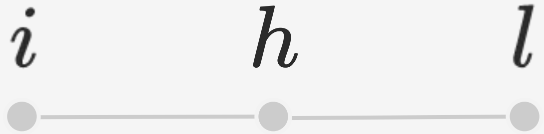


$$\left(C_h\right)_{k\to l, i\to j} = \delta_{jh}\delta_{kh}$$



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